Silesian J. Pure Appl. Math.
vol. 9, is. 1 (2019), 1–8

Wojciech BARAN[1], Krystian KURNIK[1], Wojciech NAWALANIEC[1], Albashir SHAREIF[2]

[1]Institute of Computer Science, Pedagogical University in Cracow, Poland

[2]Institute of Mathematics, Pedagogical University in Cracow, Poland

LOCAL STATIONARY HEAT FIELDS IN FIBROUS COMPOSITES

Abstract. Consider a multiply connected domain D bounded by non-overlapping circles. Introduce the complex potential $u(z) = \text{Re } \varphi(z)$ in D where the function $\varphi(z)$ is analytic in D except at infinity where $\varphi(z) \sim z$. The function $u(z)$ models the distribution of temperature in the domain D. The unknown function $\varphi(z)$ is continuously differentiable in the closures of the considered domain. We solve approximately the modified Schwarz problem when $u(z) = \text{Re } \varphi(z)$ is equal to an undetermined constant on every boundary component of D by a method of functional equations.

1. Introduction

Fibrous composites are very effective reinforcement materials used in various technical processes [1, 7]. Their excellent thermal and elastic properties attract interest of engineers. In the present paper, we discuss the stationary heat field in fibrous composites. Consider a section perpendicular to unidirectional fibers as a multiply connected domain D on the plane of variables x_1 and x_2. The axis x_3 is parallel to fibers. The heat flux in three-dimensional material is decomposed onto the linear combination of the transverse flux in the considered section and

Mathematics Subject Classification: 74A40.
Keywords: complex potential, fibrous composite, heat flux, Schwarz problem, functional equation.
Corresponding author: K. Kurnik (krystian.kurnik@up.krakow.pl).
Received: 05.05.2019.

the longitudinal flux along fibers. The longitudinal thermal flux is modeled by a one-dimensional simple flux independent in all the components of the fibrous composite. The transverse distribution of temperature is modeled by a function $u(x_1, x_2)$ harmonic in D where it satisfies Laplace's equation.

In the present paper, we apply the method of complex potentials to describe the transverse heat flux in the considered composite following the books [8] and [3]. Though analytical approximate formulae were constructed in [3], they were used to determine the effective properties of composites. The local fields were not discussed in [8] and [3]. In the case of two disks the local fields were investigated in [10] and the edges effects for the heat flux in [11]. We fill the gap of computation of local fields, first, we modify the analytical formulae from [8] and [3]. Next, we apply the modified formulae to construction of local fields. The paper essentially uses symbolic-numerical algorithms developed in [4–6] to deduce the required formulae and to construct the corresponding heat fields.

2. Method of functional equations

Let $z = x_1 + ix_2$ denote a complex variable in the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Consider non-overlapping disks $|z - a_k| < r$ $(k = 1, 2, \ldots, n)$, and the domain D, the complement of all the disks $|z - a_k| \leq r$ to $\widehat{\mathbb{C}}$. The potentials $u(z)$ is harmonic in D except at infinity where $u(z) \sim x_1 = \operatorname{Re} z$ and continuously differentiable in the closures of the considered domain. The singularity of $u(z)$ determine the external flux applied at infinity.

The distribution of temperature $u(z) \equiv u(x_1, x_2)$ is expressed through the real part of the complex potential [3, 8]:

$$u(z) = \operatorname{Re} \varphi(z), \qquad z \in D, \tag{1}$$

where $\varphi(z) = u(z) + iv(z)$ is analytic in D except at infinity where $\varphi(z) \sim z$, and continuously differentiable in the closures of the considered domain. For definiteness, we assume that the disks $|z - a_k| < r$ $(k = 1, 2, \ldots, n)$ are filled by a conductor with non-vanishing conductivity. It is worth noting that in this case the function $\varphi(z)$ is single-valued in the multiply connected domain D and does not contain logarithmic terms [8]. It follows from the fact that the divergence of the normal flux through every boundary component vanishes

$$\int_{|z-a_k|=r} \frac{\partial u}{\partial \mathbf{n}}(z)\, ds = \int_{|z-a_k|=r} \frac{\partial v}{\partial \mathbf{s}}(z)\, ds = [v]_{|z-a_k|=r} = 0,\ k = 1, 2, \ldots, n. \quad (2)$$

Here, $\frac{\partial}{\partial \mathbf{n}}$ denotes the outward unit normal derivative $\frac{\partial}{\partial \mathbf{s}}$ the tangent derivative to $|z - a_k| = r$, respectively, $[v]_{|z-a_k|=r}$ the increment of the function $v(z)$ along the circle $|z - a_k| = r$.

The gradient of $u(z)$ is related to the heat flux [3] and can be calculated by formula

$$\psi(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}, \quad (3)$$

where $\psi(z) = \varphi'(z)$ in the closure of D.

The perfect contact condition between the components is expressed by two real relations

$$u_k(z) = u(z), \quad \lambda\frac{\partial u_k}{\partial \mathbf{n}}(z) = \frac{\partial u}{\partial \mathbf{n}}(z), \quad |z - a_k| = r\ (k = 1, 2, \ldots, n), \quad (4)$$

where the conductivity of matrix is normalized to unity and the conductivity of inclusions is equal to λ. Introduce the contrast parameter

$$\varrho = \frac{\lambda - 1}{\lambda + 1}. \quad (5)$$

Two real equations (4) are reduced to the $\mathbb{R}$-linear complex condition [3]:

$$\varphi(z) = \varphi_k(z) - \varrho\, \overline{\varphi_k(z)}, \qquad |z - a_k| = r\ (k = 1, 2, \ldots, n), \quad (6)$$

where $\varphi_k(z)$ are analytic in $|z - a_k| < r$, respectively, and continuously differentiable in the closures of the considered disks. The harmonic and analytic functions are related by the equalities

$$u_k(z) = \frac{2}{\lambda + 1}\mathrm{Re}\ \varphi_k(z), \qquad |z - a_k| \le r. \quad (7)$$

Consider Schottky group of inversions and their compositions with respect to the circles $|z - a_k| = r$, $k = 1, 2, \ldots, n$ (plus the identity element):

$$z^*_{(k)} = \frac{r^2}{\overline{z - a_k}} + a_k, \quad z^*_{(k_1 k_2 \ldots, k_m)} := (z^*_{(k_2 \ldots, k_{m-1})})^*_{k_1}, \qquad (k_{j+1} \neq k_j). \quad (8)$$

Exact solution of the considered problem for any $|\varrho| < 1$ was found in the form of the absolutely and uniformly convergent Poincaré type series [3, formula (2.3.100)] up to an additive constant

$$\varphi_k(z) = z + \varrho \sum_{k_1 \neq k} \overline{z^*_{(k_1)}} + \varrho^2 \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} z^*_{(k_2 k_1)} + \varrho^3 \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} \sum_{k_3 \neq k_2} \overline{z^*_{(k_3 k_2 k_1)}} + \cdots \quad (9)$$

and

$$\varphi(z) = z + \varrho \sum_{k=1}^{n} \overline{z^*_{(k)}} + \varrho^2 \sum_{k=1}^{n} \sum_{k_1 \neq k} z^*_{(k_1 k)} + \varrho^3 \sum_{k=1}^{n} \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} \overline{z^*_{(k_2 k_1 k)}} + \cdots . \quad (10)$$

It is worth noting that our formulae (9)-(10) have simpler form than [3, formula (2.3.100)] because the restriction $|\varrho| < 1$ is supposed. The functions $\varphi_k(z)$ and $\varphi(z)$ in the limit case $\varrho = 1$ have more complicated structure, since it is represented by a uniformly and not necessary absolutely convergent Poincaré type series [3].

Below, we consider this, the most difficult in computations, case. The relation $\varrho = 1$ means that inclusions are filled by a perfectly conducting materials when λ tends to infinity. Formally, the $\mathbb{R}$-linear problem (6) does not hold in this case and has to be written as the modified Dirichlet problem [3]:

$$u_k(z) = c_k, \qquad |z - a_k| = r \ (k = 1, 2, \ldots, n), \quad (11)$$

where c_k are undetermined constants.

In the present paper, we do not consider the boundary value problem (11). We use the limit $\varrho \to 1$ in the final formulae for the local flux justified by uniform convergence of the corresponding series for $\varrho \leq 1$. Therefore, we may differentiate the corresponding uniformly convergent Poincaré type series (9)-(10) term by term and arrive at the uniformly convergent series

$$\begin{aligned} \psi_k(z) = 1 &+ \varrho \sum_{k_1 \neq k} \frac{d}{dz} \overline{z^*_{(k_1)}} + \varrho^2 \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} \frac{d}{dz} z^*_{(k_2 k_1)} \\ &+ \varrho^3 \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} \sum_{k_3 \neq k_2} \frac{d}{dz} \overline{z^*_{(k_3 k_2 k_1)}} + \cdots , \qquad z \in D_k \ (k = 1, 2, \ldots, n) \quad (12) \end{aligned}$$

and

$$\psi(z) = 1 + \varrho \sum_{k=1}^{n} \frac{d}{dz}\overline{z_{(k)}^{*}} + \varrho^2 \sum_{k=1}^{n} \sum_{k_1 \neq k} \frac{d}{dz} z_{(k_1 k)}^{*} + \varrho^3 \sum_{k=1}^{n} \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} \frac{d}{dz}\overline{z_{(k_2 k_1 k)}^{*}} + \cdots, \qquad z \in D. \quad (13)$$

It is worth noting that formula (13) is universal and takes place for $\varrho \leq 1$.

3. Implementation of series for symbolic computations

In the present section, we consider the series (13) for $\varrho = 1$. In this case, the series represents the complex flux for perfectly conducting inclusions.

We consider a sample cluster of 20000 disks simulated via random walk of disks initially placed in the regular array following the algorithm [9]. The disks are uniformly distributed in a strip of size 5×0.1. The non-overlapping uniform distribution of disks was generate in Python as list of complex numbers. Next, this list was exported to *Mathematica*. The following code for further work in *Mathematica* is presented below.

```
Nx = 20 000;
nu = 0.2;
r0 = Sqrt[nu / (Nx π)]
```

Here, the parameter nu $= 0.2$ denotes the local concentration of disks in a cluster, the radius of disks $r0 = 0.00178412$ is calculated based on nu $= 0.2$. First, we select a small square sample of size 0.1×0.1, more precisely, the square $P = \{z = x_1 + ix_2 : -0.05 < x_1, x_2 < 0.05\}$. This domain contains 433 disks.

We construct the following function Dd2 which return 0, if the point $z \notin P$ and 1 if $z \in P$.

```
Dd2[a_, z_, r_, N1_] := ∏_{m=1}^{N1} If[Abs[z - a[m]] < r, 0, 1]
```

Using (12) for $\varrho = 1$ we define the complex flux $\psi_k(z)$ (ψpSubblock) in the considered disks.

```
ψpSubblock[1, k_, z_, r_, a_, N1_] := Module[{},
  f[z] + r^2 Sum[(1 - KroneckerDelta[m, k]) 1/(z - a[m])^2 ×
      Conjugate[f[r^2/Conjugate[z - a[m]] + a[m]]], {m, 1, N1}]]
```

```
ΨpSubblock[1, z_, r_, blockFraction_] := Module[{A1, N1, a},
  neighbRadius = w2 blockFraction / 2 ;
  A1 = Select[A, Im[z] - w3 ≤ Im[#] ≤ Im[z] + w3 &];
  N1 = Length[A1]; a[m_] := A1[[m]];
  If [Dd2[a, z, r, N1] == 0, 0,
   f[z] + r^2 Sum[1/(z - a[m])^2 Conjugate[ψpSubblock[1, m,
          r^2/Conjugate[z - a[m]] + a[m], r, a, N1]], {m, 1, N1}]]]
```

Next, using (13) for $\varrho = 1$ we define the complex flux $\psi(z)$ (ΨpSubblock) in the domain D. Here, the function c returns values x and y as the real and imaginary parts of the complex number z, an argument of the function $\psi(z)$.

```
c[z_] := {Re@#, Im@#} &@ΨpSubblock[1, z, r0]
```

A list of the center coordinates (x, y) is generated. The vector field is nested between -0.05 and 0.05 for x and y values with the step 0.01. w3 is constant parameters of value 0.05 and it correspond with size of subblocks. Function f is base function, defined: f(z)=1.

```
data = Table[{{x, y}, c[x + ⅈ y]},
   {x, -0.05, 0.05, .001},
   {y, -0.05, 0.05, .001}];
```

We used a sample containing 433 disks for simulation of a random fibrous composite. A numerical approximate solution is computed and the temperature flux around the disks is constructed and displayed. The almost linear flux far away from disks is perturbed around the disks and tends to concentrate at the disks boundaries. This result can be applied in technology processes of heating to study the flux around random objects. We are going in future to compute the flux near all the 20000 random disks to get more precise results and to analyse the factors which may affect the flux distribution.

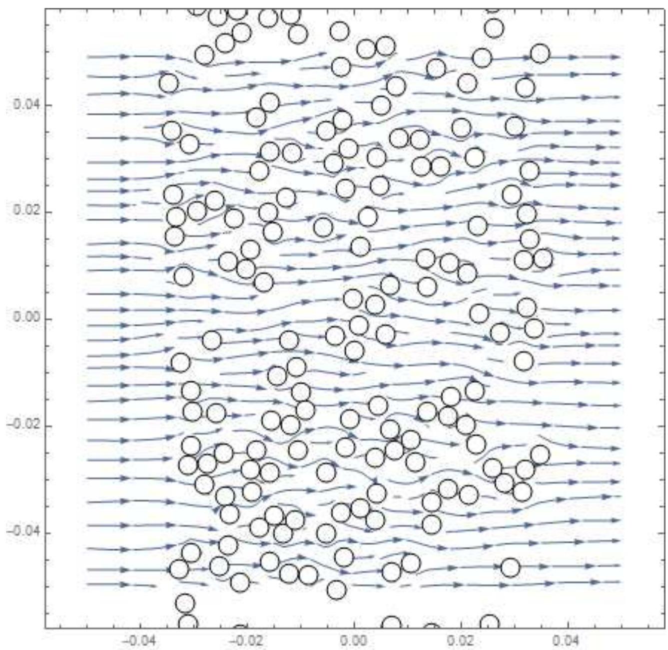

Fig. 1. The heat flux in the generated composite

Conclusion

In the present paper, we use the method of complex potentials and functional equations to determine the stationary heat flux in the fibrous composite using the uniformly and not necessary absolutely convergent Poincaré type series derived in [3] and [8]. The analytical approximate formulae from [8] and [3] are modified in order to construct the local heat flux in the considered composite. The random walk simulation are used in Python to generate a random structure. Next, the result is exported to *Mathematica* where the corresponding symbolic-numerical computations are performed and the corresponding heat flux is displayed as a vector field.

References

1. Agarwal B.D., Broutman L.J., Chandrashekhara K.: *Analysis and Performance of Fiber Composites*, 3rd edition. John Wiley, New Jersey 2006.

2. Gliński H., Grzymkowski R., Kapusta A., Słota D.: *Mathematica 8*. Jacek Skalmierski Publ., Gliwice 2012.
3. Gluzman S., Mityushev V., Nawalaniec W.: *Computational Analysis of Structured Media*. Academic Press/Elsevier, London 2017.
4. Grzymkowski R., Pleszczyński M.: *Application of the Taylor differential transformation for solving the integro-differential equations*. Silesian J. Pure Appl. Math. **8** (2018), 33–48.
5. Grzymkowski R., Pleszczyński M., Hetmaniok E.: *Application of the Taylor differential transformation in the calculus of variations*. Silesian J. Pure Appl. Math. **7** (2018), 65–82.
6. Grzymkowski R., Hetmaniok E., Pleszczyński M.: *Application of the Taylor differential transformation in the calculus of variations*. Silesian J. Pure Appl. Math. **6** (2016), 111–123.
7. Kurtyka P., Rylko N.: *Quantitative analysis of the particles distributions in reinforced composites*. Compos. Struct. **182** (2017), 412–419.
8. Mityushev V.V., Rogosin S.V.: *Constructive Methods for Linear and Nonlinear Boundary Value Problems of the Analytic Function. Theory and Applications*. Chapman & Hall/CRC, Boca Raton 2000.
9. Nawalaniec W.: *Random non-overlapping walks of disks on the plane*. Trends Math. **2** (2015), 769–775.
10. Rylko N.: *A pair of perfectly conducting disks in an external field*. Math. Model. Anal. **20** (2015), 273–288.
11. Rylko N.: *Edge effects for heat flux in fibrous composites*. Comput. Math. Appl. **70** (2015), 2283–2291.