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## APPLICATION OF THE HOMOTOPY PERTURBATION METHOD FOR THE SYSTEMS OF FREDHOLM INTEGRAL EQUATIONS

**Summary.** In this paper the convergence of homotopy perturbation method for the systems of Fredholm integral equations of the second kind is proved. Estimation of errors of approximate solutions obtained by taking the partial sum of the series is also elaborated in the paper.

## ZASTOSOWANIE HOMOTOPIJNEJ METODY PERTURBACYJNEJ DO UKŁADÓW RÓWNAŃ CAŁKOWYCH TYPU FREDHOLMA

**Streszczenie.** W artykule wykazano zbieżność homotopijnej metody perturbacyjnej dla układów równań całkowych Fredholma drugiego rodzaju. Podano także oszacowanie błędu rozwiązania przybliżonego uzyskanego jako suma częściowa tworzonego w metodzie szeregu.

2010 Mathematics Subject Classification: 65R20, 45A05.

Keywords: homotopy perturbation method, convergence, error estimation, integral equation.

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Received: 16.10.2014.

While creating this paper A. Wróbel was a student of the bachelor's degree in Mathematics.

## 1. Introduction

The homotopy perturbation method is an effective and powerful method for solving the wide class of problems [1, 6, 7, 16, 17]. In our previous papers [9, 11] we proved the convergence of homotopy perturbation method for the Fredholm and Volterra integral equations of the second kind. Moreover, the formulas for estimating the error of approximate solution were elaborated in that paper. Similar results in case of the Volterra-Fredholm integral equations of the second kind are presented in paper [8]. In the current paper we show that those previous results can be adapted for the systems of Fredholm integral equations of the second kind.

The homotopy perturbation method was already applied for solving the systems of integral equations [2, 3, 14], however in any of these papers convergence of the method or estimation of the error of approximate solution were not investigated. Only in cases of some single integral equations there exist some works (excluding papers [9, 11]) in which the authors consider convergence of the method and, eventually, estimation of the error of approximate solution. So, in paper [13] the convergence of homotopy perturbation method with the so-called convex homotopy for the Fredholm and Volterra integral equations of the second kind is discussed. Whereas, the authors of paper [4] prove the convergence and give estimation of the error of approximate solution for the piecewise homotopy perturbation method used for solving the weakly singular Volterra integral equations of the second kind. The homotopy perturbation method is a special case of the homotopy analysis method developed by Shijun Liao [5, 10, 15, 20, 21, 23].

## 2. Systems of Fredholm integral equations

We consider the system of equations of the form

$$u_i(x) - \lambda \sum_{j=1}^n \int_a^b K_{ij}(x, t) u_j(t) dt = f_i(x), \quad (1)$$

for  $i = 1, 2, \dots, n$ , where  $x \in [a, b]$ ,  $\lambda \in \mathbb{C}$ , functions  $K_{ij} \in C([a, b] \times [a, b])$  and  $f_i \in C[a, b]$  are known, whereas the functions  $u_i$  are sought. The above system of equations can be written in the matrix form

$$\mathbf{U}(x) - \lambda \int_a^b \mathbf{K}(x, y) \mathbf{U}(t) dt = \mathbf{F}(x), \quad (2)$$

where

$$\mathbf{K}(x, t) = \begin{bmatrix} K_{11}(x, t) & K_{12}(x, t) & \dots & K_{1n}(x, t) \\ K_{21}(x, t) & K_{22}(x, t) & \dots & K_{2n}(x, t) \\ \vdots & \vdots & \ddots & \vdots \\ K_{n1}(x, t) & K_{n2}(x, t) & \dots & K_{nn}(x, t) \end{bmatrix}$$

and

$$\mathbf{U}(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \\ \vdots \\ u_n(x) \end{bmatrix}, \quad \mathbf{F}(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}.$$

According to the homotopy perturbation method (for details see, for example, [11]) let us define operators  $L$  and  $N$  in the following way

$$L(\mathbf{V}) = \mathbf{V}, \quad N(\mathbf{V}) = -\lambda \int_a^b \mathbf{K}(x, t) \mathbf{V}(t) dt. \quad (3)$$

By using the above operators we obtain the homotopy operator for the system of Fredholm integral equations of the second kind

$$H(\mathbf{V}, p) = \mathbf{V}(x) - \mathbf{U}_0(x) + p \left( \mathbf{U}_0(x) - \mathbf{F}(x) - \lambda \int_a^b \mathbf{K}(x, t) \mathbf{V}(t) dt \right). \quad (4)$$

According to the method, in the next step we search for the solution of operator equation  $H(\mathbf{V}, p) = 0$  in the form of power series

$$\mathbf{V}(x) = \sum_{k=0}^{\infty} p^k \mathbf{V}_k(x), \quad (5)$$

where  $\mathbf{V}_k(x) = [v_{1,k}(x), v_{2,k}(x), \dots, v_{n,k}(x)]^T$ . In order to determine the functions  $\mathbf{V}_j$  we substitute relation (5) into equation  $H(\mathbf{V}, p) = 0$  and we get (under assumption that the series is convergent which will be discussed later):

$$\sum_{k=0}^{\infty} p^k \mathbf{V}_k(x) = \mathbf{U}_0(x) + p \left( \mathbf{F}(x) - \mathbf{U}_0(x) \right) + \sum_{k=1}^{\infty} p^k \lambda \int_a^b \mathbf{K}(x, t) \mathbf{V}_{k-1}(t) dt. \quad (6)$$

By comparing the expressions with the same powers of parameter  $p$ , we receive the relations

$$\mathbf{V}_0(x) = \mathbf{U}_0(x), \quad (7)$$

$$\mathbf{V}_1(x) = \mathbf{F}(x) - \mathbf{U}_0(x) + \lambda \int_a^b \mathbf{K}(x, t) \mathbf{V}_0(t) dt, \quad (8)$$

$$\mathbf{V}_k(x) = \lambda \int_a^b \mathbf{K}(x, t) \mathbf{V}_{k-1}(t) dt, \quad k \geq 2. \quad (9)$$

Now we proceed to discussing the convergence of series (5).

**Theorem 1.** *Let the functions  $K_{ij}$  and  $f_i$  for  $i, j \in \{1, 2, \dots, n\}$ , appearing in system (1), be continuous in regions  $\Omega_1 = [a, b] \times [a, b]$  and  $\Omega = [a, b]$ , respectively. Furthermore, as the initial approximation  $\mathbf{U}_0$  let us choose a vector of functions continuous in interval  $[a, b]$ . Certainly, it means that there exist the positive numbers  $M$  and  $N_1$  such that*

$$\|\mathbf{K}(x, t)\| \leq M \quad \wedge \quad \|\mathbf{F}(x)\| \leq N_1, \quad \text{for all } x, t \in [a, b]. \quad (10)$$

If additionally the following inequality

$$|\lambda| < \frac{1}{M(b-a)} \quad (11)$$

is satisfied, then series (5), in which the functions  $\mathbf{V}_k$  are determined by means of relations (7)–(9), is uniformly convergent in interval  $[a, b]$  for each  $p \in [0, 1]$  to the uniquely determined solution  $\mathbf{V}$ , which is a vector of functions continuous in  $[a, b]$ .

*Proof.* Let  $\mathbf{U}_0$  be a vector of functions continuous in interval  $[a, b]$ . Therefore there exists a positive number  $N_0$  such that

$$\|\mathbf{U}_0(x)\| \leq N_0, \quad \text{for all } x \in [a, b].$$

Taken assumptions imply the following estimations

$$\|\mathbf{V}_0(x)\| = \|\mathbf{U}_0(x)\| \leq N_0,$$

$$\begin{aligned} \|\mathbf{V}_1(x)\| &= \left\| \mathbf{F}(x) - \mathbf{U}_0(x) + \lambda \int_a^b \mathbf{K}(x, t) \mathbf{V}_0(t) dt \right\| \leq \\ &\leq \|\mathbf{F}(x)\| + \|\mathbf{U}_0(x)\| + |\lambda| \int_a^b \|\mathbf{K}(x, t)\| \|\mathbf{V}_0(t)\| dt \leq \\ &\leq N_1 + N_0 + |\lambda| \int_a^b M N_0 dt = N_0 + N_1 + |\lambda| M N_0 (b-a) =: B, \end{aligned}$$

$$\begin{aligned} \|\mathbf{V}_2(x)\| &= \left\| \lambda \int_a^b \mathbf{K}(x, t) \mathbf{V}_1(t) dt \right\| \leq |\lambda| \int_a^b \|\mathbf{K}(x, t)\| \|\mathbf{V}_1(t)\| dt \leq \\ &\leq |\lambda| \int_a^b M B dt = B |\lambda| M (b-a), \end{aligned}$$

where  $B := N_0 + N_1 + |\lambda| M N_0 (b-a)$ . In general we have

$$\|\mathbf{V}_k(x)\| \leq B |\lambda|^{k-1} M^{k-1} (b-a)^{k-1}, \quad x \in [a, b], \quad k \geq 1.$$

In this way, for considered series (5) we get for  $p \in [0, 1]$ :

$$\sum_{k=0}^{\infty} p^k \mathbf{V}_k(x) \leq \sum_{k=0}^{\infty} \|\mathbf{V}_k(x)\| \leq N_0 + \sum_{k=1}^{\infty} B |\lambda|^{k-1} M^{k-1} (b-a)^{k-1}.$$

The last series in the above estimation is the convergent geometric series possessing the common ratio  $q = |\lambda| M (b-a) < 1$  (we remember assumption (11)). Hence, the discussed series (5) is uniformly convergent in interval  $[a, b]$  for each  $p \in [0, 1]$  to continuous function  $\mathbf{V}$ . As it results from considerations included in [12, 18, 22] the received solution is unique.  $\square$

**Remark 2.** Similar result as in Theorem 1 holds true in the class of square integrable functions.

**Remark 3.** Construction of the method implies that the sum of series (5) for  $p = 1$  satisfies system (2). Under assumptions of Theorem 1 the series (5) for  $p = 1$  is convergent to the unique solution of system (2), independently on the selected initial approximation  $\mathbf{U}_0$ , if only  $\|\mathbf{U}_0(x)\| \leq N_0$  for all  $x \in [a, b]$ .

**Remark 4.** In presented theorem the interval  $[a, b]$  can be replaced by intervals  $(a, b)$ ,  $(a, b]$  or  $[a, b)$ , whereas the condition of continuity of functions  $K_{ij}$  and  $f_i$  in the appropriate regions  $\Omega_1$  and  $\Omega$  must be strengthened by the additional assumption of boundedness of these functions. Moreover, the conditions  $K_{ij} \in C([a, b] \times [a, b])$  or  $\|K_{ij}\| \leq M$  can be replaced by some weaker conditions, for example, by the Lebesgue integrability of  $K_{ij}$  on the set  $[a, b] \times [a, b]$  and by the inequality (see [19]):

$$\int_a^b \|\mathbf{K}(x, t)\| dt \leq M (b-a)$$

for the respective norm of the matrix kernel  $\mathbf{K}$ .

If we are not able to determine the sum of series (5) (for  $p = 1$ ), then as the approximate solution of considered equation we can accept the partial sum of this series. If we take the first  $n + 1$  components, we obtain the so-called  $n$ -order approximate solution

$$\hat{\mathbf{U}}_n(x) = \sum_{k=0}^n \mathbf{V}_k(x). \quad (12)$$

Now let us proceed to estimating the error of approximate solution constructed in this way.

**Theorem 5.** *Error of the  $n$ -order approximate solution can be estimated in the following way*

$$E_n \leq B \frac{(|\lambda| M (b-a))^n}{1 - |\lambda| M (b-a)}, \quad (13)$$

where  $E_n := \sup_{x \in [a, b]} \|\mathbf{U}(x) - \widehat{\mathbf{U}}_n(x)\|$ ,  $B := N_0 + N_1 + |\lambda| M N_0 (b-a)$  and the constants  $M$ ,  $N_1$  and  $N_0$  are such that

$$\|\mathbf{K}(x, t)\| \leq M \quad \wedge \quad \|\mathbf{F}(x)\| \leq N_1 \quad \wedge \quad \|\mathbf{U}_0(x)\| \leq N_0 \quad \forall x, t \in [a, b]. \quad (14)$$

*Proof.* By using the estimations of functions  $\mathbf{V}_k$  we get for any  $x \in [a, b]$ :

$$\begin{aligned} \|\mathbf{U}(x) - \widehat{\mathbf{U}}_n(x)\| &= \left\| \sum_{k=0}^{\infty} \mathbf{V}_k(x) - \sum_{k=0}^n \mathbf{V}_k(x) \right\| = \left\| \sum_{k=n+1}^{\infty} \mathbf{V}_k(x) \right\| \leq \\ &\leq \sum_{k=n+1}^{\infty} \|\mathbf{V}_k(x)\| \leq B \sum_{k=n+1}^{\infty} |\lambda|^{k-1} M^{k-1} (b-a)^{k-1} = B \frac{(|\lambda| M (b-a))^n}{1 - |\lambda| M (b-a)}. \end{aligned}$$

□

### 3. Example

In the example we use the discussed method for solving the following system of Fredholm integral equations of the second kind

$$\begin{aligned} u_1(x) &= -\frac{1}{12} x^3 + \frac{11}{12} x^2 + \frac{29}{120} x - \frac{9}{80} + \\ &\quad + \frac{1}{4} \left( \int_0^1 (x-t)^3 u_1(t) dt + \int_0^1 (x-t)^2 u_2(t) dt \right), \\ u_2(x) &= \frac{35}{48} x^3 + \frac{361}{240} x^2 + \frac{53}{80} x + \frac{13}{168} + \\ &\quad + \frac{1}{4} \left( \int_0^1 (x-t)^2 u_1(t) dt + \int_0^1 (x-t)^3 u_2(t) dt \right). \end{aligned}$$

Solution of the above system is given by the functions

$$u_{d1}(x) = x^2, \quad u_{d2}(x) = x^3 + x^2 + x.$$

If as the vector norm we take

$$\|\mathbf{V}\|_{\infty} := \max_{1 \leq k \leq n} |V_k|,$$

then it induces the matrix norm of the form

$$\|\mathbf{A}\|_{\infty} := \max_{1 \leq k \leq n} \sum_{j=1}^n |a_{kj}|.$$

Thus it is easy to notice that  $\|\mathbf{K}(x, t)\|_{\infty} \leq 2$  for every  $(x, t) \in [0, 1]^2$ , that is  $M = 2$ . The same result can be obtained equally easy if we take

$$\|\mathbf{V}\|_1 := \sum_{ki=1}^n |V_k| \quad \text{and} \quad \|\mathbf{A}\|_1 := \max_{1 \leq j \leq n} \sum_{k=1}^n |a_{kj}|.$$

It means that for the considered system of integral equations the condition (11) is satisfied which implies the convergence of homotopy perturbation method.

By taking the zero initial approximation  $\mathbf{U}_0(x) = (0, 0)^T$  and next by applying relations (7)–(9) we get successively

$$\begin{aligned} \mathbf{V}_0(x) &= \mathbf{U}_0(x) = (0, 0)^T, \\ \mathbf{V}_1(x) &= \left( -\frac{9}{80} + \frac{29x}{120} + \frac{11x^2}{12} - \frac{x^3}{12}, \frac{13}{168} + \frac{53x}{80} + \frac{361x^2}{240} + \frac{35x^3}{48} \right)^T, \\ \mathbf{V}_2(x) &= \left( \frac{713}{6300} - \frac{69x}{280} + \frac{7699x^2}{80640} + \frac{211x^3}{2880}, \right. \\ &\quad \left. -\frac{5281}{67200} + \frac{137867x}{403200} - \frac{41351x^2}{80640} + \frac{22021x^3}{80640} \right)^T, \\ &\vdots \end{aligned}$$

As the approximate solution  $\widehat{\mathbf{U}}_n = (\widehat{u}_{1,n}, \widehat{u}_{2,n})^T$  defined by partial sum (12) for  $n = 5$  we receive

$$\begin{aligned} \widehat{u}_{1,5}(x) &= -1.1532 \cdot 10^{-7} + 2.75134 \cdot 10^{-7} x + 1 \cdot x^2 - 2.65244 \cdot 10^{-8} x^3, \\ \widehat{u}_{2,5}(x) &= 8.04464 \cdot 10^{-8} + 1 \cdot x + 1 \cdot x^2 + 1 \cdot x^3, \end{aligned}$$

whereas for  $n = 15$  we get

$$\begin{aligned} \widehat{u}_{1,15}(x) &= 2.72971 \cdot 10^{-15} - 1.4225 \cdot 10^{-14} x + 1 \cdot x^2 - 1.8247 \cdot 10^{-14} x^3, \\ \widehat{u}_{2,15}(x) &= -4.13206 \cdot 10^{-15} + 1 \cdot x + 1 \cdot x^2 + 1 \cdot x^3. \end{aligned}$$

All calculations were executed with the aid of computational software *Mathematica*.

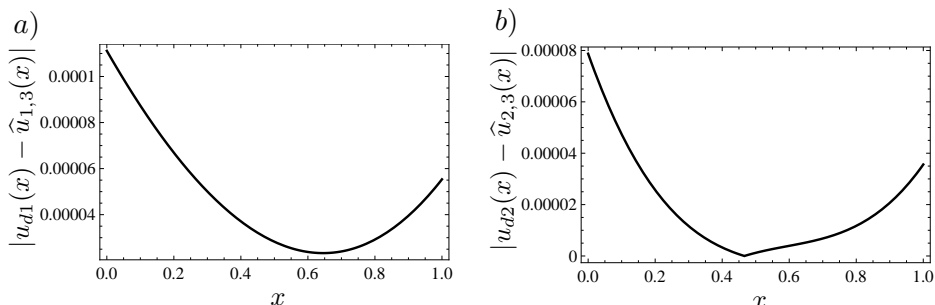
In Table 1 there are presented the errors ( $\|u_{di} - \widehat{u}_{i,n}\| = \sup_{x \in [0,1]} |u_{di}(x) - \widehat{u}_{i,n}(x)|$ ) which occur in approximating the exact solution by the successive approximate solutions. Whereas, distributions of error in the entire interval  $[0, 1]$  for

$n = 3$  and  $n = 8$  are displayed in Figures 1 and 2. Presented results indicate that the method is fast convergent and computing just a few (a dozen or so) first terms of the series ensures a very good approximation of the exact solution.

Table 1

Errors of the exact solution approximations

| $n$ | $\ u_{d1} - \hat{u}_{1,n}\ $ | $\ u_{d2} - \hat{u}_{2,n}\ $ | $n$ | $\ u_{d1} - \hat{u}_{1,n}\ $ | $\ u_{d2} - \hat{u}_{2,n}\ $ |
|-----|------------------------------|------------------------------|-----|------------------------------|------------------------------|
| 1   | 0.1125                       | $7.7381 \cdot 10^{-2}$       | 6   | $2.3015 \cdot 10^{-9}$       | $3.5532 \cdot 10^{-9}$       |
| 2   | $2.0164 \cdot 10^{-3}$       | $3.1473 \cdot 10^{-3}$       | 7   | $1.2465 \cdot 10^{-10}$      | $8.5761 \cdot 10^{-11}$      |
| 3   | $1.1114 \cdot 10^{-4}$       | $7.8733 \cdot 10^{-5}$       | 8   | $2.5543 \cdot 10^{-12}$      | $3.9478 \cdot 10^{-12}$      |
| 4   | $2.1328 \cdot 10^{-6}$       | $3.2904 \cdot 10^{-6}$       | 9   | $1.3863 \cdot 10^{-13}$      | $9.4478 \cdot 10^{-14}$      |
| 5   | $1.1532 \cdot 10^{-7}$       | $8.0446 \cdot 10^{-8}$       | 10  | $2.8905 \cdot 10^{-15}$      | $4.4708 \cdot 10^{-15}$      |

Fig. 1. Distribution of error of the exact solution approximation for  $n = 3$ 

Rys. 1. Rozkład błędów rozwiązania przybliżonego dla  $n = 3$

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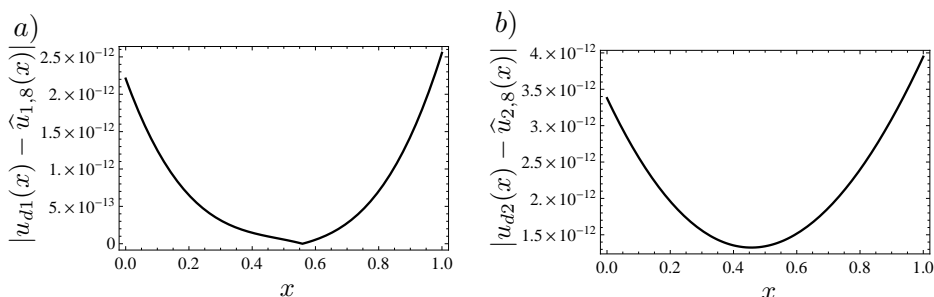


Fig. 2. Distribution of error of the exact solution approximation for  $n = 8$

Rys. 2. Rozkład błędów rozwiązania przybliżonego dla  $n = 8$

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