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APPLICATION OF THE ANALYTIC-NUMERICAL METHOD IN SOLVING THE PROBLEM WITH MOVING BOUNDARY

Summary. The paper presents a method of the analytic-numerical nature applied for finding the approximate solutions of the selected class of problems which can be reduced to the one-phase solidification problem of a plate with the unknown a priori, varying in time boundary of the region in which the solution is sought. Presented method is attractive from the engineer's point of view since it is relatively easy for using and does not require either sophisticated numerical techniques or far advanced mathematical tools.

ZASTOSOWANIE ANALITYCZNO–NUMERYCZNEJ METODY DO ROZWIĄZANIA ZAGADNIENIA Z RUCHOMYM BRZEGIEM

Streszczenie. W pracy przedstawiono metodę o analityczno–numerycznym charakterze zastosowaną do przybliżonego rozwiązywania wybranej klasy problemów, które można sprowadzić do jednofazowego zagadnienia krzepnięcia płyty z nieznaną a priori, zmienną w czasie granicą obszaru, w którym poszukiwane jest rozwiązanie. Prezentowana metoda jest atrakcyjna z inżynierskiego punktu widzenia, gdyż jest stosunkowo łatwa w użyciu i nie wymaga stosowania wyszukanych technik numerycznych ani zaawansowanych narzędzi matematycznych.

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1. Introduction

Planning of the technology for producing the ingots in course of continuous casting process is a complicated and multistage problem. Effectiveness of designed technology is valued by means of the quality of continuous ingot. One of the most important factors influencing this quality is the field of temperature in the solidifying metal volume, defined by an important parameter for the considered technology which is the location of freezing front determining the thickness of solidified layer (thickness of the ingot skin). Too fast either too slow increase of the solidified layer is unacceptable. If the skin of solidified ingot leaving the casting mould will be too thin then it can break and the liquid metal can leak which may cause a very serious damage of the continuous casting equipment. From the other hand, too fast increase of the skin is connected with the excessive drop of temperature on its cross-section which may cause the high thermal tension leading to the ingot cracking.

Considering these facts we will discuss the flat ingot of the rectangular cuboid shape, produced in the vertical continuous casting equipment with the constant velocity of the ingot forming. Dimensions of its cross-section sides $2\overline{x}$ and $2\overline{y}$ ($2\overline{x}$ – thickness of the ingot, $2\overline{y}$ – width of the ingot) satisfy condition $2\overline{x} \ll 2\overline{y}$. This assumption enables to consider the solidifying ingot as an axisymmetrical 2-dimensional object in which the thermal processes take place in the surface of thermal symmetry (see Figure 1).

Moreover, if we assume that the ingot is produced from the metal solidifying in constant temperature T^* and in such temperature it is poured into the casting mould, then the non-failure working of the continuous casting equipment generates the pseudo-steady field of temperature in the solidified part of the ingot of length $\overline{z}$ which, in the coordinate system oriented in space like in Figure 1, is described with the aid of equation

$$v\frac{\partial T}{\partial z} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right), \quad \varphi(z) < x < \overline{x}, \quad 0 < z \leqslant \overline{z}, \tag{1}$$

where $T = T(x, z)$ denotes the temperature, v – the velocity vector coordinate in the direction of ingot forming, a a is the thermal diffusivity coefficient and $\varphi(z)$ defines a function describing the freezing front location

$$\varphi(z) = \overline{x} - \xi(z), \tag{2}$$

where $\xi(z)$ denotes the ingot skin thickness (thickness of the solidified layer) variable on the ingot length and

$$\xi\,(0) = 0. \tag{3}$$

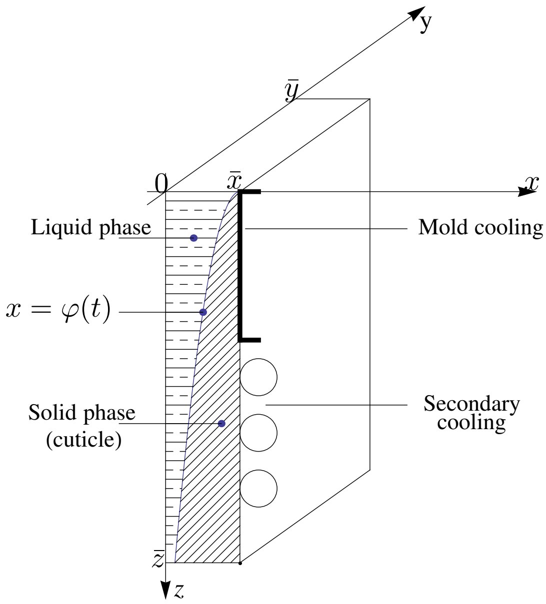

Fig. 1. The modelled area
Rys. 1. Modelowany obszar

Because of taken assumptions equation (1) is complemented by the boundary conditions on the freezing front

$$\lambda \mathrm{grad}\, T|_{x=\varphi(z)} \cdot \mathbf{n} = \gamma\kappa v_n, \quad 0 \leqslant z \leqslant \overline{z}, \tag{4}$$

$$T|_{x=\varphi(z)} = T^*, \quad 0 \leqslant z \leqslant \overline{z}, \tag{5}$$

where $\boldsymbol{n}$ denotes the unitary vector directed outside and normal to the freezing front, as well as by one of the conditions defined on the ingot surface

$$T|_{x=\overline{x}} = f, \quad 0 < z \leqslant \overline{z}, \tag{6}$$

or

$$-\lambda \frac{\partial T}{\partial x}\bigg|_{x=\overline{x}} = q, \quad 0 < z \leqslant \overline{z} \tag{7}$$

or, relatively

$$-\lambda \frac{\partial T}{\partial x}\bigg|_{x=\overline{x}} = \alpha\left(T|_{x=\overline{x}} - T^{\infty}\right), \quad 0 < z \leqslant \overline{z}. \tag{8}$$

In the above equations symbols T^{∞}, λ, γ and κ denote the ambient temperature, thermal conductivity coefficient, metal density and latent heat, respectively. Whereas elements $f = f(z)$, $q = q(z)$, $\alpha = \alpha(z)$ and $v_n = v_n(z)$ define, in turn, temperature of the ingot surface, distribution of the heat flux, distribution of the heat transfer coefficient and coordinate of the velocity vector of the freezing front moving in direction normal to this front.

One can simplify more equation (1) when the ingot is produced from the material with the low value of thermal conductivity coefficient. It is because, in this case, the thermal conductivity in direction of the ingot forming is usually small, therefore it can be neglected [7]. Taking it into consideration one can ignore therm $\frac{\partial^2 T}{\partial z^2}$ in equation (1). In result of this, in place of elliptic equation (1) we receive the parabolic equation

$$v\frac{\partial T}{\partial z} = a\frac{\partial^2 T}{\partial x^2}, \quad \varphi(z) < x < \overline{x}, \quad 0 < z \leqslant \overline{z}, \tag{9}$$

in which the variable z plays the role of time. Whereas, boundary condition (4) on the freezing front takes the form

$$-\lambda\frac{\partial T}{\partial x}\bigg|_{x=\varphi(z)} = \gamma\kappa v_n, \quad 0 \leqslant z \leqslant \overline{z}. \tag{10}$$

2. Approach to the problem

Mathematical modeling of thermal processes, combined with the reversible phase transitions of type: liquid phase – solid phase, leads to the moving boundary problems. Solving of such determined problem requires in most cases to use the appropriate numerical techniques. In the current paper we present the approximate analytic-numerical method, especially attractive from the engineer's point of view because of its respective simplicity. Proposed method is based on two elements. The first one is the known formalism of initial expansion of the sought function, describing the field of temperature, into the power series in which some number of coefficients is determined with the aid of boundary conditions. Second element consists in approximating the function defining the freezing front location by means of the broken line, parameters of which are numerically determined.

As we have mentioned above, method of solving the problem formulated in previous section is based, in the first step, on the proper presentation of the function representing the expected solution in the form of power series, similarly as it was done in papers [1–6, 8]. In considered case the series is of the following form

$$T\left(x,z\right)=\sum_{i=0}^{\infty}\frac{A_i(z)(x-\overline{x}+\xi(z))^i}{i!}, \tag{11}$$

where $A_i(z)$ denote the unknown, dependent on variable z , functional coefficients. In case of the elliptic problem these coefficients will be determined by using equation (1), condition (5) and transformed condition (4) which can be written in form

$$\lambda\left(\left.\frac{\partial T}{\partial x}\right|_{x=\varphi(z)}-\varphi'(z)\left.\frac{\partial T}{\partial z}\right|_{x=\varphi(z)}\right)=\gamma\kappa v\varphi'(z),\quad 0\leqslant z\leqslant\overline{z}. \tag{12}$$

Whereas, in case of the parabolic problem we use equation (9) and conditions (5) and (10) on the freezing front.

Assumed form (11) of the sought solution and relation (2) imply that

$$\frac{\partial T}{\partial x}=\sum_{i=0}^{\infty}\frac{A_{i+1}(z)\left(x-\overline{x}+\xi(z)\right)^i}{i!}, \tag{13}$$

$$\frac{\partial T}{\partial z}=\sum_{i=0}^{\infty}\left(A_i'\left(z\right)+A_{i+1}(z)\xi'(z)\right)\frac{\left(x-\overline{x}+\xi(z)\right)^i}{i!}, \tag{14}$$

$$\frac{\partial^2 T}{\partial x^2}=\sum_{i=0}^{\infty}\frac{A_{i+2}(z)\left(x-\overline{x}+\xi(z)\right)^i}{i!}, \tag{15}$$

$$\begin{aligned}\frac{\partial^2 T}{\partial z^2}=\sum_{i=0}^{\infty}\Bigg(&A_i''(z)+2A_{i+1}'(z)\xi'(z)+A_{i+1}(z)\xi''(z)+\\&+A_{i+2}(z)\left(\xi'(z)^2\right)\Bigg)\frac{(x-\overline{x}+\xi(z)^i)}{i!}.\end{aligned} \tag{16}$$

By substituting properly the received formulas into equation (1) we obtain

$$\begin{aligned}&v\sum_{i=0}^{\infty}(A_i'(z)+A_{i+1}(z)\xi'(z))\frac{\left(x-\overline{x}+\xi(z)\right)^i}{i!}=\\&\qquad=a\sum_{i=0}^{\infty}\big(A_{i+2}(z)+A_i''(z)+2A_{i+1}'(z)\xi'(z)+\\&\qquad+A_{i+1}(z)\xi''(z)+A_{i+2}(z)\left(\xi'(z)\right)^2\big)\frac{\left(x-\overline{x}+\xi(z)\right)^i}{i!}.\end{aligned} \tag{17}$$

And, by substituting the same formulas into equation (9), we get the relation

$$\begin{aligned} v\sum_{i=0}^{\infty}(A_i'(z)+A_{i+1}(z)\xi'(z))\frac{(x-\overline{x}+\xi(z))^i}{i!} &= \\ = a\sum_{i=0}^{\infty}A_{i+2}(z)\frac{(x-\overline{x}+\xi(z))^i}{i!}&. \end{aligned} \tag{18}$$

Comparing the terms situated on the both sides of equations (17) and (18), preceding expressions $\frac{(x-\overline{x}+\xi(z))^i}{i!}$, $i=0,1,2,\ldots$, we receive for the elliptic problem

$$\begin{aligned} a\bigg(A_i''(z)+2A_{i+1}'(z)\xi'(z)+A_{i+1}(z)\xi''(z)+A_{i+2}(z)\left(1+(\xi'(z))^2\right)\bigg) &= \\ = v(A_i'(z)+A_{i+1}(z)\xi'(z)), \quad i=0,1,2,\ldots& \end{aligned} \tag{19}$$

and for the parabolic problem

$$v(A_i'(z)+A_{i+1}(z)\xi'(z)) = aA_{i+2}(z), \quad i=0,1,2,\ldots. \tag{20}$$

In case of the elliptic problem, it follows from conditions (2), (5) and (12) that

$$A_0(z)=T^*, \tag{21}$$

$$A_1(z) = -\frac{\gamma\kappa v\xi'(z)}{\lambda\left((\xi'(z))^2+1\right)}. \tag{22}$$

Since we have coefficients $A_0(z)$ and $A_1(z)$ we are able, by using formula (19), to determine the remaining coefficients $A_i(z)$, $i=2,3,4,\ldots$ Thus we get

$$\begin{aligned} A_{i+2}(z) = \frac{v\left(A_i'(z)+A_{i+1}(z)\xi'(z)\right)}{a\left(1+(\xi'(z))^2\right)} &+ \\ -\frac{\left(A_i''(z)+2A_{i+1}'(z)\xi'(z)+A_{i+1}(z)\xi''(z)\right)}{1+(\xi'(z))^2}&, \end{aligned} \tag{23}$$

for $i=0,1,2,\ldots$. From the obtained formulas for coefficients $A_i(z)$, $i=0,1,2,\ldots$, we can conclude that all the coefficients $A_i(z)$, $i=1,2,3,\ldots$, except coefficient $A_0(z)$, depend on the still unknown function $\xi(z)$, its derivatives and powers of those derivatives. One can try to determine analytically this function by using one of conditions (6), (7) or (8). In particular, for condition (6) we receive

$$\sum_{i=0}^{\infty}A_i(z)\frac{(\xi(z))^i}{i!} = f(z). \tag{24}$$

However, equation (24) is so much complicated that determination of function $\xi(z)$ with the aid of this equation is possible only in case of its certain simplification. In particular, by taking only two first terms of the series in relation (24) we obtain differential equation of the form

$$T^* - \frac{\gamma\kappa v \xi'(z)\xi(z)}{\lambda\left((\xi'(z))^2+1\right)} = f(z). \tag{25}$$

Since function $\xi(z)$ is increasing by assumption, after simple transformations relation (25) implies that

$$\xi'(z) = \frac{\gamma\kappa v\xi(z) + \sqrt{(\gamma\kappa v\xi(z))^2 - 4\lambda^2(f(z)-T^*)^2}}{2\lambda\left(T^* - f(z)\right)}. \tag{26}$$

Unluckily, an analytic solution of equation (26) is not possible for arbitrarily given function $f(z)$.

Problem of determining function $\xi(z)$ is even more complicated if we consider the boundary conditions of the second (7) and third (8) kind. For finding the analytic solution in these cases, similar necessary simplifications must be made.

For example, considering condition (7) of the second kind we get the equation

$$-\lambda \sum_{i=0}^{\infty} \frac{A_{i+1}(z)(\xi(z))^i}{i!} = q(z). \tag{27}$$

By taking only one (the first) term of the series in relation (27) we obtain

$$\frac{\gamma\kappa v\xi'(z)}{(\xi'(z))^2+1} = q(z). \tag{28}$$

Hence, after simple transformations it results that

$$\xi'(z) = \frac{\gamma\kappa v + \sqrt{(\gamma\kappa v)^2 - 4(q(z))^2}}{2q(z)}. \tag{29}$$

Equation (29), similarly as equation (26), will have the explicit solution only if function $q(z)$ will have the appropriate form.

Considering the parabolic equation, similar relations can be received. In this case, conditions (2), (5) and (10) imply that

$$A_0(z) = T^*, \tag{30}$$

$$A_1(z) = -\frac{\gamma\kappa v}{\lambda}\xi'(z). \tag{31}$$

Analogically as in the previous case, by having the coefficients $A_0(z)$ and $A_1(z)$ we can use formula (20) we can calculate the remaining coefficients $A_i(z)$, $i = 2, 3, 4, \ldots$ We obtain

$$A_{i+2}(z) = \frac{v}{a}\left(A_i'(z) + A_{i+1}(z)\xi'(z)\right), \quad i = 0, 1, 2, \ldots \tag{32}$$

In here as well, all the coefficients $A_i(z)$, $i = 2, 3, 4, \ldots$, except coefficient $A_0(z)$, depend on the still unknown function $\xi(z)$, its derivatives and powers of those derivatives.

Whereas the equations connecting function $\xi(z)$ with functions defining the heat transfer on the ingot surface remain unchanged with accuracy to coefficients $A_i(z)$, $i = 1, 2, 3, \ldots$. In particular, for boundary condition (6) of the first kind we get

$$\sum_{i=0}^{\infty} A_i(z)\frac{(\xi(z))^i}{i!} = f(z), \tag{33}$$

for boundary condition (7) of the second kind we have

$$-\lambda\sum_{i=0}^{\infty}\frac{A_{i+1}(z)(\xi(z))^i}{i!} = q(z) \tag{34}$$

and finally for boundary condition (8) of the third kind we obtain

$$-\lambda\sum_{i=0}^{\infty}\frac{A_{i+1}(z)(\xi(z))^i}{i!} = \alpha(z)\left(\sum_{i=0}^{\infty}\frac{A_i(z)(\xi(z))^i}{i!} - T^{\infty}\right). \tag{35}$$

Received equations, similarly like in case of the elliptic problem, are so much complicated that the analytic determination of function $\xi(z)$ with the aid of those is possible only with some limitations and by applying appropriate simplifications.

In particular, by taking only three first terms in the series from relation (33) we get

$$A_0(z) + A_1(z)\xi(z) + \frac{1}{2}A_2(z)\xi^2(z) = f(z). \tag{36}$$

From this, by using formulas (30)–(32) we obtain the following differential equation

$$T^* - \frac{\gamma\kappa v}{\lambda}\xi'(z)\xi(z) - \frac{\gamma\kappa v^2}{2a\lambda}\left(\xi'(z)\xi(z)\right)^2 = f(z). \tag{37}$$

In result of solving this equation under the assumption of satisfying condition (3) we receive finally

$$\xi(z) = \sqrt{\frac{2}{\gamma\kappa v}\int_0^z\left(\sqrt{(a\gamma\kappa)^2 + 2a\lambda\gamma\kappa(T^* - f(\tau))} - \frac{a}{v}\right)d\tau}. \tag{38}$$

Although in some cases, like for example when $T = T^0 = constans$, $0 \leqslant z \leqslant \overline{z}$, we can evaluate from equation (38) very simple formula

$$\xi(z) = \sqrt{\frac{2}{\gamma\kappa v}\left(\sqrt{(a\gamma\kappa)^2 + 2a\lambda\gamma\kappa(T^* - T^0)} - \frac{a}{v}\right) z}, \tag{39}$$

in general case relation (38) is not very suitable since we are not always able to calculate the integral appearing in this relation.

We can obtain an approximate solution of this problem, as well as of the other problems, with no difficulties, if we assume that function defining the freezing front location is approximated by the broken line (see Figure 2), it means

$$\varphi(z) = \overline{x} - \overline{\xi}(z), \tag{40}$$

where

$$\overline{\xi}(z) = x_j + m_j(z - z_j), \quad z \in \left(z_j, z_{j+1}\right\rangle, \quad j = 0, 1, 2, \ldots, \tag{41}$$

and we have

$$z_0 = 0, \quad z_{j+1} > z_j, \quad j = 0, 1, 2, \ldots \tag{42}$$

and

$$x_0 = 0, \quad x_j = m_{j-1}\left(z_j - z_{j-1}\right) + x_{j-1}, \quad j = 1, 2, 3, \ldots, \tag{43}$$

where parameters m_j, $j = 0, 1, 2, \ldots$, will be determined numerically.

Set assumption implies that $\xi(z) = \overline{\xi}(z)$, thus

$$\xi'(z) = m_j, \quad z \in \left(z_j, z_{j+1}\right\rangle, \quad j = 0, 1, 2, \ldots \tag{44}$$

From formulas (21) and (22) and relation (23) we get

$$A_0(z) = T^*, \quad z \in \left(z_j, z_{j+1}\right\rangle, \quad j = 0, 1, 2, \ldots, \tag{45}$$

$$A_1(z) = -\frac{\gamma\kappa v m_j}{\lambda\left(m_j^2 + 1\right)}, \quad z \in \left(z_j, z_{j+1}\right\rangle, \quad j = 0, 1, 2, \ldots, \tag{46}$$

$$A_{i+2}(z) = \frac{-\gamma\kappa (v m_j)^{i+2}}{\lambda a^{i+1} (m_j^2 + 1)^{i+2}}, \quad z \in \left(z_j, z_{j+1}\right\rangle, \quad i = 0, 1, 2, \ldots, \quad j = 0, 1, 2, \ldots \tag{47}$$

which means that function (11), describing the temperature field in the solid phase, can be presented in form

$$T(x, z) = T^* - \frac{\gamma\kappa a}{\lambda}\left(\exp\left(\frac{v m_j\left(x - \overline{x} + x_j + m_j\left(z - z_j\right)\right)}{a\left(m_j^2 + 1\right)}\right) - 1\right), \tag{48}$$

for $z \in \langle z_j, z_{j+1} \rangle$ and $j = 0, 1, 2, \ldots$ One can be easily verify that function (48) satisfies conditions (4) and (5) and its unknown elements are only parameters m_j, $j = 0, 1, 2, \ldots$, for calculation of which one of the boundary conditions (6)–(8) will be used.

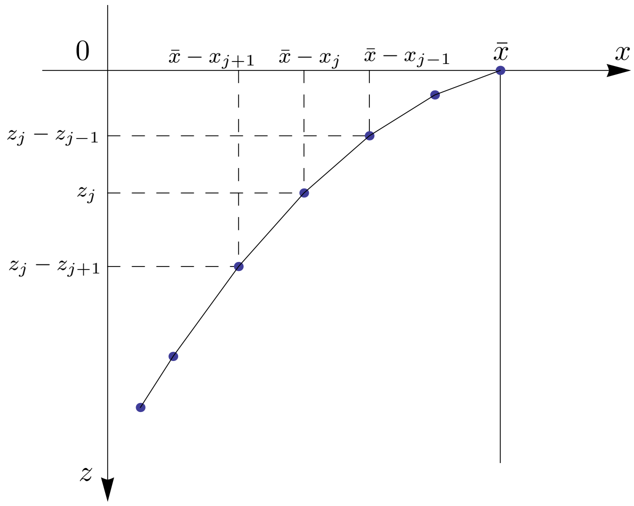

Fig. 2. Approximation of the function describing the freezing front location
Rys. 2. Aproksymacja funkcji określającej położenie granicy rozdziału faz

If we require that for each z_{j+1}, $j = 0, 1, 2, \ldots$, one of the boundary conditions (6), (7) or (8) is satisfied, we receive the equation enabling to determine the sought parameters. In particular, by applying condition (6) we have

$$f\left(z_{j+1}\right) = T^* - \frac{\gamma \kappa a}{\lambda} \left(\exp\left(\frac{v m_j \left(x_j + m_j \left(z_{j+1} - z_j\right)\right)}{a\left(m_j^2 + 1\right)} \right) - 1 \right), \qquad (49)$$
$$j = 0, 1, 2, \ldots$$

By proceeding in similar way we can also determine the forms of function defining the freezing front location for the conditions of second (7) and third (8) kind. In particular, for condition (7) of the second kind we have

$$q\left(z_{j+1}\right) = \frac{\gamma \kappa v m_j}{m_j^2 + 1} \exp\left(\frac{v m_j \left(x_j + m_j \left(z_{j+1} - z_j\right)\right)}{a\left(m_j^2 + 1\right)} \right), \qquad (50)$$
$$j = 0, 1, 2, \ldots,$$

whereas, for condition (8) of the third kind we get

$$\frac{\gamma\kappa v m_j}{m_j^2+1}\exp\left(\frac{v m_j(x_j+m_j\left(z_{j+1}-z_j\right))}{a\left(m_j^2+1\right)}\right)=\alpha\left(z_{j+1}\right)\times$$
$$\times\left(T^*-T^\infty-\frac{\gamma\kappa a}{\lambda}\left(\exp\left(\frac{v m_j\left(x_j+m_j\left(z_{j+1}-z_j\right)\right)}{a\left(m_j^2+1\right)}\right)-1\right)\right), \quad (51)$$
$$j=0,1,2,\ldots$$

For each of the conditions (6)–(8) parameters m_j, $j=0,1,2,\ldots$, are described by means of equations which cannot be solved analytically. Equations (49)-(51) can be solved by applying one of the many known methods of determining the approximate roots of nonlinear equations, like for example the direct iteration method.

3. Numerical examples

Example 1

Let us consider the theoretical example in which the function of temperature $T=T(x,z)$ is expressed by means of equation $T\left(x,z\right)=1000-\exp\left(x+0,2z-\overline{x}\right)$ and function describing the freezing front location is of the form $\varphi\left(z\right)=\overline{x}-\xi\left(z\right)$ in which the reconstructed, variable on the ingot length, thickness of the ingot skin $\xi(z)$ is defined by relation $\xi(z)=0,2z$. Values of the other parameters are the following: thickness of material of the solidifying plate $\overline{x}=2$, density $\gamma=4$, thermal conductivity coefficient $\lambda=1$, latent heat $\kappa=130$, solidification temperature $T^*=999$, ambient temperature $T^\infty=50$ and we assume that the heat transfer with environment is described by means of one of conditions (6)–(8). Moreover, let us assume that we consider the solidification process until the moment of time t^∞=100. For the material data, environment parameters and duration time chosen in such way, the function $T(x,z)$ satisfies equation (1) as well as condition (4) and (5) on the freezing front.

By having the function describing the field of temperature in solid phase one can generate, by using boundary conditions (6)–(8), the functions defining: temperature of the ingot surface $f(z)$, heat flux density $q(z)$ of the heat derived outside and heat transfer coefficient $\alpha(z)$.

Another important parameter is the discretization density of variable z. In this paper we take that the considered interval $\langle 0,\bar{z}\rangle$ is evenly divided into m sections of length $\Delta z=\bar{z}/m$ which means that $z_j=j\Delta z$, for $j=0,1,2,\ldots,m$.

The main object, investigated in testing calculations, is the precision of reconstruction of function $\xi(z)$ describing the thickness of solidified layer variable in time.

Since in considered example the reconstructed function is linear and is approximated by linear function, we will present briefly the efficiency of discussed method by confining ourselves to the boundary condition of the third kind only. By using formula (51) for given parameters characterizing the discussed problem and by taking $m = 30$ nodes of partition of the time interval we received the result presented graphically in Figure 3.

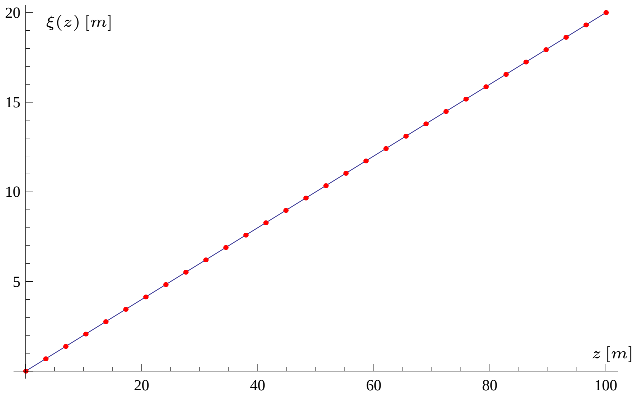

Fig. 3. Plot of function describing the thickness of solidified layer $\xi(z)$ varying in time (solid line) and its approximation $\overline{\xi}(z)$ (points) obtained for $m = 30$ (for boundary condition of the third kind)

Rys. 3. Wykres funkcji określającej zmienną w czasie grubość warstwy zakrzepłej $\xi(z)$ (linia ciągła) i jej przybliżenia $\overline{\xi}(z)$ (punkty) dla $m = 30$ (dla warunku brzegowego III rodzaju)

In Figure 4 the absolute errors of obtained approximate solution are displayed. The errors are determined according formula

$$\Delta\left(z_j\right) = \left|\xi\left(z_j\right) - \overline{\xi}(z_j)\right|, \quad j = 1, 2, ..., m, \tag{52}$$

where $\Delta\left(z_j\right)$ denotes the absolute error, $\xi\left(z_j\right)$ is the exact value, $\overline{\xi}(z_j)$ is the approximate value of function describing the thickness of solidified layer varying in time and z_j represents the j-th node of discretization of variable z for $j = 1, 2, ..., m$.

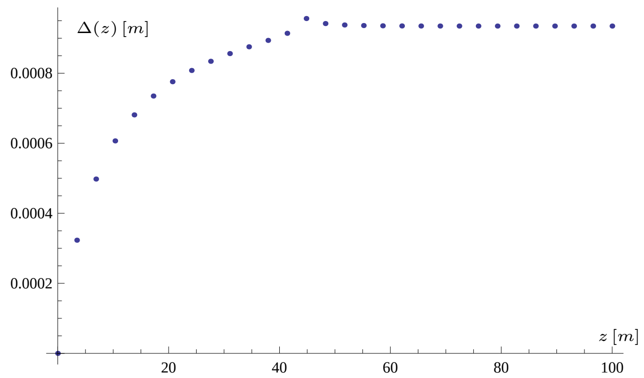

Fig. 4. Distribution of absolute errors for $m = 30$ (for boundary condition of the third kind)

Rys. 4. Rozkład błędów bezwzględnych dla $m = 30$ (dla warunku brzegowego III rodzaju)

Example 2

To confirm effectiveness of investigated method we present now the results of further research in which we assume that the plate of thickness $2\overline{x} = 0,2$ $[m]$ is casted, material of which is specified by the following parameters: density $\gamma = 7000$ $[kg/m^3]$, thermal conductivity coefficient $\lambda = 25$ $[W/mK]$, specific heat $c = 800$ $[J/kgK]$, latent heat $\kappa = 247$ $[kJ/kg]$, solidification temperature $T^* = 1500$ $[K]$, ambient temperature $T^\infty = 320$ $[K]$, velocity vector coordinate in the direction of ingot forming $v = 0,8$ $[m/min]$ and the heat transfer with environment is defined by means of one of boundary conditions (6)–(8). Additionally, let us assume that we consider the process of ingot solidification until the moment of time in which the ingot reaches the length $\bar{z} = vt^*$. Now, similarly like in [1], we set that the field of temperature in solid phase is described with the aid of function of form (11), under the assumption that the reconstructed thickness of the ingot skin $\xi(z)$, varying along the ingot length, is expressed by one of the formulas:

$$\xi(z) = \xi_1(z) = 0.001z^2, \tag{53}$$

$$\xi(z) = \xi_2(z) = 0.00001z^2, \tag{54}$$

$$\xi(z) = \xi_3(z) = 0.1\sin\left(\frac{\pi z}{20}\right), \tag{55}$$

in which $\bar{z} = 10$ $[m]$.

Let us discuss the case of the first kind condition (6). By using formula (49) for the given parameters characterizing the considered problem and by taking $m = 30$

nodes of interval $\langle 0, \bar{z} \rangle$ discretization, we receive the result displayed graphically in Figure 5, where the solid line denotes the exact solution $\xi_1(z)$, whereas the approximate solution $\overline{\xi}_1(z)$ is designated by points. In Figure 6 the absolute errors (52) of this approximation are showed.

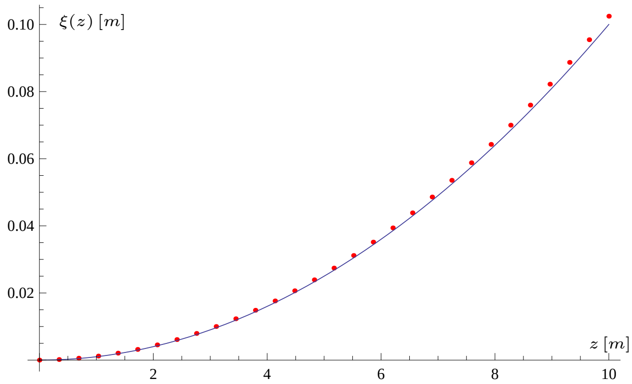

Fig. 5. Graph of function describing the thickness of solidified layer $\xi_1(z)$, varying in time (solid line) and its approximation $\overline{\xi}_1(z)$ (points) calculated for $m = 30$ (for boundary condition of the first kind)

Rys. 5. Wykres funkcji określającej zmienną w czasie grubość warstwy zakrzepłej $\xi_1(z)$ (linia ciągła) i jej przybliżenia $\overline{\xi}_1(z)$ (punkty) dla $m = 30$ (dla warunku brzegowego I rodzaju)

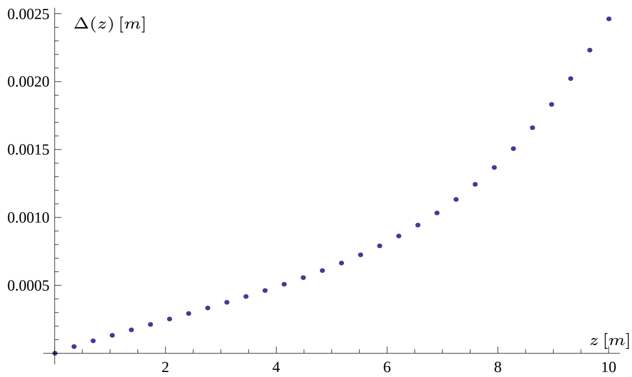

Fig. 6. Distribution of absolute errors for $m = 30$ (for boundary condition of the first kind)

Rys. 6. Rozkład błędów bezwzględnych dla $m = 30$ (dla warunku brzegowego I rodzaju)

Example 3

Let us consider now the case of boundary condition (7) of the second kind for the problem specified in Example 2. By applying formula (50) for the assumed parameters characterizing the discussed problem and by taking $m = 30$ nodes of discretization, we get the result presented in Figure 7, with the absolute errors displayed in Figure 8.

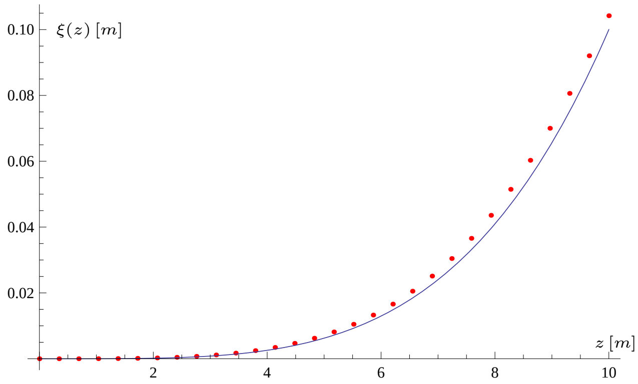

Fig. 7. Graph of function describing the thickness of solidified layer $\xi_2(z)$, varying in time (solid line) and its approximation $\overline{\xi}_2(z)$ (points) calculated for $m = 30$ (for boundary condition of the second kind)

Rys. 7. Wykres funkcji określającej zmienną w czasie grubość warstwy zakrzepłej $\xi_2(z)$ (linia ciągła) i jej przybliżenia $\overline{\xi}_2(z)$ (punkty) dla $m = 30$ (dla warunku brzegowego II rodzaju)

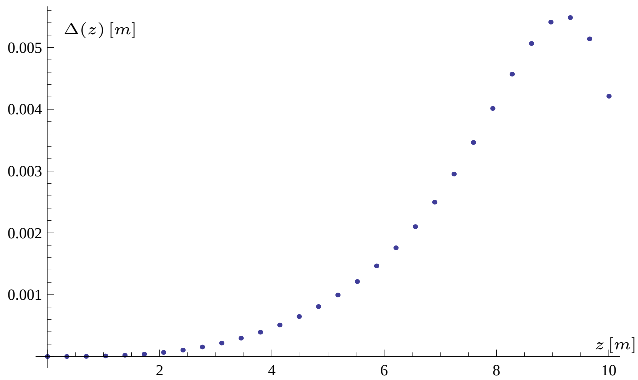

Fig. 8. Distribution of absolute errors for $m = 30$ (for boundary condition of the second kind)

Rys. 8. Rozkład błędów bezwzględnych dla $m = 30$ (dla warunku brzegowego II rodzaju)

Example 4

Finally, we take into consideration the case of boundary condition (8) of the third kind for the problem characterized in Example 2. Similarly like in previous cases, but by using formula (51), for the given parameters specifying the discussed problem any by taking $m = 30$ nodes of the time interval discretization we receive the result presented in Figure 9, with the absolute errors showed in Figure 10.

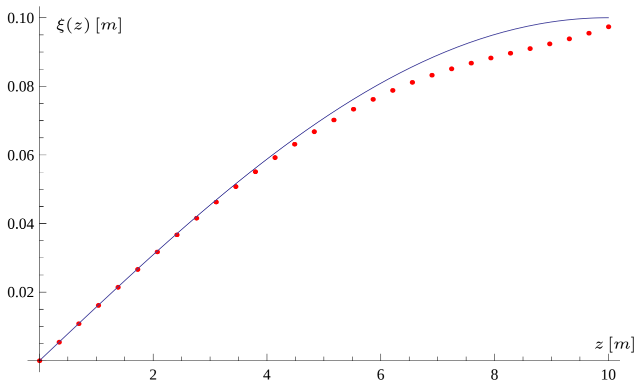

Fig. 9. Graph of function describing the thickness of solidified layer $\xi_3(z)$, varying in time (solid line) and its approximation $\overline{\xi}_3(z)$ (points) calculated for $m = 30$ (for boundary condition of the third kind)

Rys. 9. Wykres funkcji określającej zmienną w czasie grubość warstwy zakrzepłej $\xi_3(z)$ (linia ciągła) i jej przybliżenia $\overline{\xi}_3(z)$ (punkty) dla $m = 30$ (dla warunku brzegowego III rodzaju)

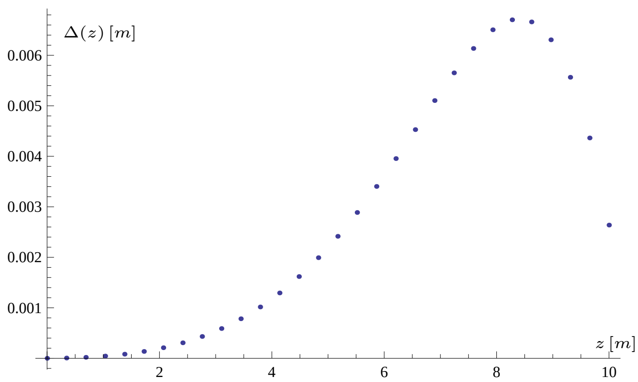

Fig. 10. Distribution of absolute errors for $m = 30$ (for boundary condition of the third kind)

Rys. 10. Rozkład błędów bezwzględnych dla $m = 30$ (dla warunku brzegowego III rodzaju)

4. Conclusions

The paper presents the method of analytic-numerical nature used for determining the approximate solution of the selected kind of problems reducible to the one-phase solidification problem of a plate with the unknown a priori, varying in time boundary of the region in which the solution is sought. Proposed method is based on two elements: on the expansion of the sought function, describing the temperature field, into the power series, some coefficients of which are determined by using the boundary conditions and on the approximation of the function, defining the location of freezing front, with the broken line, parameters of which are determined numerically. Effectiveness and usefulness of the approach have been illustrated with examples.

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Omówienie

W artykule przedstawiono atrakcyjną z inżynierskiego punktu widzenia metodę rozwiązywania wybranej klasy problemów, które można sprowadzić do jednofazowego zagadnienia krzepnięcia płyty z nieznaną a priori, zmienną w czasie granicą obszaru, w którym poszukiwane jest rozwiązanie. Metoda ta stanowi połączenie technik analitycznych oraz numerycznych i bazuje na znanym formalizmie wstępnego rozwinięcia poszukiwanej funkcji, opisującej pole temperatury, w szereg potęgowy, którego pewne współczynniki wyznaczane są z warunków brzegowych. Zaprezentowane rozwiązania zilustrowane zostały przykładami.