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ON THE RIEMANN INTEGRAL OF SET-VALUED FUNCTIONS

Summary. In the paper we present a generalization of the Riemann integral of set-valued function introduced by Dinghas and independently by Hukuhara to the case of nonempty bounded closed convex subsets of Banach spaces. Moreover, we compare properties of this integral to the Riemann integral of real function and consider its relation with the Hukuhara derivative.

CAŁKA RIEMANNA Z FUNKCJI WIELOWARTOŚCIOWYCH

Streszczenie. Praca zawiera uogólnienie definicji całki Riemanna z funkcji wielowartościowej wprowadzonej przez Dinghasa i niezależnie przez Hukuharę na przypadek niepustych domkniętych, ograniczonych i wypukłych podzbiorów przestrzeni Banacha. Następnie porównujemy własności tak zdefiniowanej całki z całką Riemanna z funkcji rzeczywistej oraz rozważamy jej związek z pochodną Hukuhary.

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1. Introduction

The concept of Riemann integral of set-valued functions was introduced by Dinghas and Hukuhara, independently, in fifties and sixties. However, the authors considered only set-valued functions with values being nonempty compact and convex subsets of euclidean spaces. The main goal of this paper is a generalization of their definitions to the case of nonempty bounded closed and convex subsets of Banach spaces, especially of infinite dimensions.

Our work is organized in the following way. In Section 2 we introduce some definitions and notations that will be needed in the rest of the paper. There are also given two theorems devoted to the equivalent conditions of integrability of set-valued functions. The next section contains some properties of integrals defined in this way, again especially in the case of infinite dimensional spaces. Finally, in Section 4, we propose to look at the Riemann integral of set-valued functions from a viewpoint of Hukuhara derivatives.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space and let $clb(X)$ denote the set of all nonempty convex closed bounded subsets of X. For two subsets $A, B \in clb(X)$, we set $A + B = \{a + b : a \in A, b \in B\}$, $\lambda A = \{\lambda a : a \in A\}$ for $\lambda \geqslant 0$ and $A \overset{*}{+} B = cl(A + B) = cl(clA + clB)$, where clA means the closure of A in X. It is easy to see that $(clb(X), \overset{*}{+}, \cdot)$ satisfies the following properties

$$\lambda(A \overset{*}{+} B) = \lambda A \overset{*}{+} \lambda B,\ (\lambda + \mu)A = \lambda A \overset{*}{+} \mu A,\ \lambda(\mu A) = (\lambda\mu)A,\ 1 \cdot A = A$$

for each $A, B \in clb(X)$ and $\lambda \geqslant 0,\ \mu \geqslant 0$. If $A, B, C \in clb(X)$, then the equality $A \overset{*}{+} C = B \overset{*}{+} C$ implies $A = B$ (see e.g. [2, Theorem II-17, p. 48]). Thus the cancellation law holds in $clb(X)$ with the operation $\overset{*}{+}$.

The set $clb(X)$ is a metric space with the Hausdorff metric h defined by

$$h(A, B) = \inf\{t > 0 : A \subset B + tS, B \subset A + tS\},$$

where S is the closed unit ball in X. The metric space $(clb(X), h)$ is complete (see e.g. [2, Theorem II-3, p. 40]). Moreover, the Hausdorff metric h is translation invariant since

$$h(A \overset{*}{+} C, B \overset{*}{+} C) = h(A + C, B + C) = h(A, B)$$

and positively homogeneous, i.e.,

$$h(\lambda A, \lambda B) = \lambda h(A, B)$$

for all $\lambda \geqslant 0$ and $A, B, C \in clb(X)$ (cf. [1, Lemma 2.2]).

Let F be a set-valued function defined on an interval $[a, b]$ with values in $clb(X)$. A set $\Delta = \{x_0, x_1, \ldots, x_n\}$, where $a = x_0 < x_1 < \ldots < x_n = b$, is called a *partition* of $[a, b]$. For given partition Δ we put $\delta(\Delta) := \max\{x_i - x_{i-1} : \ i \in \{1, \ldots, n\}\}$ and form the approximating sum

$$S(\Delta, \tau) = (x_1 - x_0)F(\tau_1) \overset{*}{+} \ldots \overset{*}{+} (x_n - x_{n-1})F(\tau_n),$$

where τ is a system $(\tau_1, \ldots, \tau_n)$ of intermediate points ($\tau_i \in [x_{i-1}, x_i]$). A sequence (Δ^ν) of partitions is called *normal* when $\lim_{\nu\to\infty} \delta(\Delta^\nu) = 0$. If for each normal sequence (Δ^ν, τ^ν), with Δ^ν – partitions of $[a, b]$ and τ^ν, $\nu \in \mathbb{N}$ – systems of intermediate points, the sequence of the approximating sums $(S(\Delta^\nu, \tau^\nu))$ always tends to the same limit $I \in clb(X)$, then F is said to be *Riemann integrable* on $[a, b]$ and $\int_a^b F(x)dx := I$.

The Riemann integral for set-valued function with compact convex values was investigated by A. Dinghas [3] and M. Hukuhara [4]. In paper [6] the above integral was introduced and some properties for the case $F\colon [a, b] \to clb(X)$ were studied.

Theorem 1. *$I \in clb(X)$ is the Riemann integral of $F\colon [a, b] \to clb(X)$ on $[a, b]$ if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$h\left(I, S(\Delta, \tau)\right) < \varepsilon \tag{1}$$

holds for each partition $\Delta = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\delta(\Delta) < \delta$ and each system of intermediate points $\tau_i \in [x_{i-1}, x_i]$.

Proof. Sufficiency. Let us choose an arbitrary normal sequence (Δ^ν) of partitions of $[a, b]$ and an arbitrary sequence of systems τ^ν of intermediate points. For each $\varepsilon > 0$ one may find $\delta > 0$ such that the condition (1) holds for each partition Δ with diameter $\delta(\Delta) < \delta$. Since the sequence (Δ^ν) is normal, there is $\nu_0 \in \mathbb{N}$ such that $\delta(\Delta^\nu) < \delta$ for all $\nu > \nu_0$. On account of (1) we obtain

$$h\left(I, S(\Delta^\nu, \tau^\nu)\right) < \varepsilon$$

for any $\nu > \nu_0$. So the Riemann integral of F exists and is equal to I.

Necessity. There is a number $\varepsilon > 0$ such that for every $\delta > 0$ there is a partition Δ of $[a, b]$ with diameter $\delta(\Delta) < \delta$ and a system τ of intermediate points corresponding to Δ for which

$$h\left(\int_a^b I(x)dx, S(\Delta,\tau)\right) \geqslant \varepsilon.$$

Taking $\delta = \frac{1}{\nu}$, where $\nu \in \mathbb{N}$, one may find a sequence (Δ^ν) of partitions of $[a,b]$ and corresponding sequence (τ^ν) of systems of intermediate points such that $\delta(\Delta^\nu) < \frac{1}{\nu}$ and

$$h\left(\int_a^b I(x)dx, S(\Delta^\nu,\tau^\nu)\right) \geqslant \varepsilon,$$

which contradicts the fact that I is a Riemann integral of F. $\square$

In a similar way one may prove the following theorem which gives us two conditions equivalent to the integrability of a set–valued function $F\colon [a,b] \to clb(X)$.

Theorem 2. *Let $F\colon [a,b] \to clb(X)$ be a set-valued function. Then the following conditions are equivalent*

(i) *F is integrable on $[a,b]$;*

(ii) *for every $\varepsilon > 0$ one may find a $\delta > 0$ in such a way that for each partition Δ of $[a,b]$ with diameter $\delta(\Delta) < \delta$, its subpartition Δ' and arbitrary systems τ, τ' of intermediate points corresponding to Δ and Δ', respectively, an inequality*

$$h\left(S(\Delta,\tau), S(\Delta',\tau')\right) < \varepsilon \tag{2}$$

holds;

(iii) *for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all partitions Δ^1, Δ^2 of $[a,b]$ with diameters $\delta(\Delta^k) < \delta$ and systems τ^k, $k \in \{1,2\}$, of intermediate points corresponding to Δ^k, respectively, an inequality*

$$h\left(S(\Delta^1,\tau^1), S(\Delta^2,\tau^2)\right) < \varepsilon \tag{3}$$

holds.

Let us denote

$$\|A\| := h\left(A, \{0\}\right), \qquad A \in clb(X).$$

Then we may define:

Definition 3. A set–valued function $F\colon [a,b] \to clb(X)$ is called *bounded*, if one may find $M > 0$ such that $\|F(x)\| \leqslant M$ for all $x \in [a,b]$.

In the sequel we will denote

$$\sum_{i=1}^{n}{}^{*} A_i := A_1 \overset{*}{+} \ldots \overset{*}{+} A_n.$$

Remark 4. *If a set–valued function $F\colon [a,b] \to clb(X)$ is integrable, then it is bounded.*

Proof. Let us suppose that F is unbounded and let us denote $M(x_1) = \|F(x_1)\|$, where $x_1 = a$. Next we choose $x_2 \in [a,b]$ such that

$$M(x_2) = \|F(x_2)\| > \max\{1, M(x_1)\}.$$

By the inductive way one may find a sequence (x_n), $x_n \in [a,b]$ for which

$$M(x_{n+1}) = \|F(x_{n+1})\| > \max\{n, M(x_n)\}.$$

From (x_n) we can choose a Cauchy subsequence (x_{p_n}). Let x be its limit.

Let us take any $\varepsilon > 0$ and choose $\delta > 0$ in the same way as in Theorem 2 (ii). It follows that for a fixed partition $\Delta = \{y_0, \ldots, y_m\}$ with diameter smaller than δ, its subpartition Δ' and any systems of intermediate points τ, τ' connected with Δ and Δ' respectively, the following inequality holds

$$h\left(S(\Delta,\tau), S(\Delta',\tau')\right) < \varepsilon. \tag{4}$$

From (x_{p_n}) one may choose monotonic subsequence (u_n). So there exists $k \in \{1, \ldots, m\}$ such that almost all u_n belong to $[y_{k-1}, y_k]$. Since the sequence $(\|F(u_n)\|)$ is strictly increasing, the sequence (u_n) must be strictly monotonic. So we may assume that $u_n \in (y_{k-1}, y_k)$, $n \in \mathbb{N}$. Let

$$\begin{aligned} S_n = (y_1 - y_0)F(y_0) \overset{*}{+} \ldots \overset{*}{+} (y_{k-1} - y_{k-2})F(y_{k-2}) \overset{*}{+} (u_n - y_{k-1})F(y_{k-1}) \overset{*}{+} \\ \overset{*}{+} (y_k - u_n)F(u_n) \overset{*}{+} (y_{k+1} - y_k)F(y_k) \overset{*}{+} \ldots \overset{*}{+} (y_m - y_{m-1})F(y_{m-1}) \end{aligned}$$

and

$$S(\Delta) = (y_1 - y_0)F(y_0) \overset{*}{+} \ldots \overset{*}{+} (y_m - y_{m-1})F(y_{m-1}).$$

Hence

$$\begin{aligned}\varepsilon > h\left(S_n, S(\Delta)\right) = \\ &= h\Big((y_k - u_n)F(u_n) \overset{*}{+} (u_n - y_{k-1})F(y_{k-1}), (y_k - y_{k-1})F(y_{k-1})\Big) = \\ &= h\left((y_k - u_n)F(u_n), (y_k - u_n)F(y_{k-1})\right) = (y_k - u_n)h\left(F(u_n), F(y_{k-1})\right).\end{aligned}$$

Since

$$\|F(u_n)\| = h\left(F(u_n), \{0\}\right) \leqslant h\left(F(u_n), F(y_{k-1})\right) + \|F(y_{k-1})\|,$$

we have

$$\varepsilon > (y_k - u_n)\left(\|F(u_n)\| - \|F(y_{k-1})\|\right).$$

If $x \neq y_k$, then the right-hand side of the above inequality tends to $+\infty$, what leads to a contradiction.

If $x = y_k$, then S_n can be defined by

$$\begin{aligned}S_n = (y_1 - y_0)F(y_0) \overset{*}{+} \ldots \overset{*}{+} (y_{k-1} - y_{k-2})F(y_{k-2}) \overset{*}{+} (u_n - y_{k-1})F(u_n) \overset{*}{+} \\ \overset{*}{+} (y_k - u_n)F(y_k) \overset{*}{+} (y_{k+1} - y_k)F(y_k) \overset{*}{+} \ldots \overset{*}{+} (y_m - y_{m-1})F(y_{m-1})\end{aligned}$$

and

$$\begin{aligned}\varepsilon > h\left(S_n, S(\Delta)\right) = \\ &= h\Big((u_n - y_{k-1})F(u_n) \overset{*}{+} (y_k - u_n)F(y_k), (y_k - y_{k-1})F(y_{k-1})\Big).\end{aligned}$$

Since

$$\begin{aligned}\|(u_n - y_{k-1})F(u_n) \overset{*}{+} (y_k - u_n)F(y_k)\| \leqslant \\ &\leqslant h\Big((u_n - y_{k-1})F(u_n) \overset{*}{+} (y_k - u_n)F(y_k), (y_k - y_{k-1})F(y_{k-1})\Big) + \\ &\qquad + \|(y_k - y_{k-1})F(y_{k-1})\|,\end{aligned}$$

we have

$$\varepsilon > \|(u_n - y_{k-1})F(u_n) \overset{*}{+} (y_k - u_n)F(y_k)\| - \|(y_k - y_{k-1})F(y_{k-1})\|,$$

what yields

$$\varepsilon > (u_n - y_{k-1})\|F(u_n)\| - (y_k - u_n)\|F(y_k)\| - (y_k - y_{k-1})\|F(y_{k-1})\|$$

for all $n \in \mathbb{N}$. Then the right-hand side of the above inequality tends to $+\infty$ and in this case we also obtain a contradiction. $\square$

3. Properties of the Riemann integral

Now we would like to present some properties of the integral defined in the previous section. In the sequel a set-valued function $F\colon [a,b] \to clb(X)$ will be called *continuous*, if it is continuous with respect to the Hausdorff metric h.

Theorem 5. *Each continuous set-valued function $F\colon [a,b] \to clb(X)$ is integrable.*

Next we will consider behaviour of integrals of set-valued functions in case of subintervals of $[a,b]$.

Theorem 6. *If $[c,d] \subset [a,b]$ and a set-valued function $F\colon [a,b] \to clb(X)$ is integrable on $[a,b]$, then F is integrable on $[c,d]$.*

We also have:

Theorem 7. *If a set-valued function $F\colon [a,b] \to clb(X)$ is integrable on $[a,c]$ and $[c,b]$, then it is integrable on its sum $[a,b]$.*

Moreover, we claim that:

Theorem 8. *If a set-valued function $F\colon [a,b] \to clb(X)$ is integrable in the Riemann sense on $[a,b]$ and $c \in (a,b)$, then*

$$\int_a^b F(x)\,dx \;=\; \int_a^c F(x)\,dx \;\stackrel{*}{+}\; \int_c^b F(x)\,dx. \tag{5}$$

All proofs of these theorems go with very similar patterns. For reader's convenience we will prove the last one. The rest of proofs may be found in [7, pp. 14–18].

Proof of Theorem 8. On account of Theorem 6, F is integrable on intervals $[a,c]$ and $[c,b]$. Let us fix a positive integer p and take $\Delta_x^p = \{x_0, x_1, \ldots, x_p\}$, $\Delta_y^p = \{y_0, y_1, \ldots, y_p\}$, where $x_0 = a$, $x_p = c = y_0$, $y_p = b$ and $x_i - x_{i-1} = \frac{c-a}{p}$, $y_i - y_{i-1} = \frac{b-c}{p}$, $i \in \{1, \ldots, p\}$. Moreover, suppose that $\tau_x^p = \{x_1, \ldots, x_p\}$ and $\tau_y^p = \{y_1, \ldots, y_p\}$. Clearly, $\Delta^p = \Delta_x^p \cup \Delta_y^p$ is a partition of $[a,b]$, and $\tau^p = \tau_x^p \cup \tau_y^p$ is a corresponding system of intermediate points. One may notice that

$$S(\Delta^p, \tau^p) = S(\Delta_x^p, \tau_x^p) \stackrel{*}{+} S(\Delta_y^p, \tau_y^p) \tag{6}$$

and

$$\lim_{p\to\infty} \delta(\Delta^p_x) = \lim_{p\to\infty} \delta(\Delta^p_y) = \lim_{p\to\infty} \delta(\Delta^p) = 0.$$

All integrals in (5) exist, so taking $p \to \infty$ in (6), we obtain (5). □

The following theorem also gives us a counterpart of the well–known result for the Riemann integral of real function. But in case of real functions a stronger result may be obtained. In the sequel we will come back to this problem.

Theorem 9. *Let set-valued functions $F, G : [a, b] \to clb(X)$ and a real function $h(F, G) : [a, b] \to [0, \infty)$ be integrable in the Riemann sense on $[a, b]$. Then*

$$h\left(\int_a^b F(x)\, dx, \int_a^b G(x)\, dx\right) \leqslant \int_a^b h(F(x), G(x))\, dx. \tag{7}$$

Proof. For an arbitrary natural p let us take a partition $\Delta^p = \{x_0, x_1, \dots, x_p\}$ such that $x_0 = a$, $x_p = b$ and $x_i - x_{i-1} = \frac{b-a}{p}$ for all $i \in \{1, \dots, p\}$ and $\tau^p = \{x_1, \dots, x_p\}$. Now let us consider Riemann sums for F and G:

$$S_F(\Delta^p, \tau^p) = (x_1 - x_0)F(x_1) \overset{*}{+} \dots \overset{*}{+} (x_p - x_{p-1})F(x_p)$$

and

$$S_G(\Delta^p, \tau^p) = (x_1 - x_0)G(x_1) \overset{*}{+} \dots \overset{*}{+} (x_p - x_{p-1})G(x_p).$$

Since the Hausdorff metric is translation invariant and positively homogeneous, we obtain

$$h\left(S_F(\Delta^p, \tau^p), S_G(\Delta^p, \tau^p)\right) \leqslant \sum_{i=1}^{p} (x_i - x_{i-1}) h\left(F(x_i), G(x_i)\right). \tag{8}$$

Since $\lim_{p\to\infty} \delta(\Delta^p) = 0$, taking $p \to \infty$, (8) implies our assertion. □

In case of real functions from the integrability of functions $f, g \colon [a, b] \to \mathbb{R}$ it follows that $|f - g|$ is an integrable function. In case of set-valued function the situation is quite different and the integrability of F and G does not imply that $h(F, G)$ is also integrable as the following example shows.

Example 10. Let us suppose that Y is a space of all bounded functions on $[0, 1]$. Then Y with a norm

$$\|x\| = \sup_{t\in[0,1]} |x(t)|,$$

is a real Banach space.

Let $V \subset (0,1]$ be an immeasurable set (with respect to the Lebesgue measure) and let us define a set–valued function $G\colon [0,1] \to 2^Y$ by

$$G(t) = \begin{cases} \{x \mid x(u) = 0,\ u \in [0,1]\}, & \text{if} \quad t \in [0,1] \setminus V, \\ \{x \mid x(u) = 0,\ u \in [0,1] \setminus \{t\},\ x(t) \in [1,2]\}, & \text{if} \quad t \in V. \end{cases}$$

Let us notice that $G(t)$ are compact and convex subsets of Y. Let us donote by Θ a function equal to 0 on $[0,1]$. We want to show that

$$h\left(\{\Theta\}, S(\Delta,\tau)\right) \leqslant 4\delta(\Delta) \tag{9}$$

for each partition $\Delta = \{t_0 = 0, t_1, \ldots, t_n = 1\}$ of an interval $[0,1]$ and for each system of intermediate points $\tau = \{\tau_1, \ldots, \tau_n\}$, $\tau_k \in [t_{k-1}, t_k]$. Since G has compact values,

$$S(\Delta,\tau) = \sum_{k=1}^{n} \Delta t_k \, G(\tau_k),$$

where $\Delta t_k = t_k - t_{k-1}$.

Let us fix an arbitrary $x \in S(\Delta,\tau)$ and $t \in [0,1]$. Since $0 \notin V$, $x(0) = 0$. Now we want to find an estimation of $|x(t)|$ for $t \in (0,1]$. Clearly, there is $k \in \{1, \ldots, n\}$ for which $t \in (t_{k-1}, t_k]$. Let us notice that

- if $t \neq \tau_k$ and $t \neq \tau_{k+1}$, then $x(t) = 0$;
- if $\tau_k \neq \tau_{k+1}$ and $t = \tau_k$ (or $t = \tau_{k+1}$), then $x(t) = 0$ for $t \notin V$ or $x(t) \in \Delta t_k[1,2]$ ($x(t) \in \Delta t_{k+1}[1,2]$) for $t \in V$;
- if $t = t_k = \tau_k = \tau_{k+1}$, then $x(t) = 0$ for $t \notin V$ or $x(t) \in (\Delta t_k + \Delta t_{k+1})[1,2]$ for $t \in V$.

In each case $|x(t)| \leqslant 4\delta(\Delta)$, $t \in [0,1]$, so $\|x\| \leqslant 4\delta(\Delta)$, and (9) holds. From (9) it follows that G is an integrable set-valued function and moreover,

$$\int_0^1 G(t)dt = \{\Theta\}.$$

Let $F\colon [0,1] \to clb(Y)$ be define by a formula $F(t) = \{\Theta\}$, $t \in [0,1]$. Therefore F is integrable as a constant set-valued function.

A mapping $[0,1] \ni t \mapsto h\left(F(t), G(t)\right) \in [0,\infty)$ takes only two values. Namely, 0 for $t \notin V$ and 2 for $t \in V$. On account of definition of V the mapping $h\left(F(t), G(t)\right)$ is immeasurable with respect to the Lebesgue measure, so it is integrable neither in Lebesgue nor in the Riemann sense.

We finish this section proving that the counterpart of antiderivative in case of set-valued mappings is lipschitzian.

Theorem 11. *Let $F\colon [a,b] \to clb(X)$ be integrable on $[a,b]$. Then the mapping*

$$[a,b] \ni x \mapsto \int_a^x F(t)dt \in clb(X)$$

is lipschitzian.

Proof. Let $x, y \in [a,b]$. It suffices to consider $x \neq y$, so one may assume that $x < y$. Then

$$h\left(\int_a^x F(t)dt, \int_a^y F(t)dt\right) = h\left(\int_a^x F(t)dt, \int_a^x F(t)dt \overset{*}{+} \int_x^y F(t)dt\right) = \\ = h\left(\{0\}, \int_x^y F(t)dt\right).$$

Since on account of Remark 4 F is bounded, there is $M > 0$ such that $h(\{0\}, F(t)) \leqslant M$ for any $t \in [a,b]$. For each normal sequence of partitions Δ^ν of $[x,y]$ and systems $\tau^\nu$of intermediate points we have

$$h\left(\{0\}, \int_x^y F(t)dt\right) = h\left(\{0\}, \lim_{\nu\to\infty} S(\Delta^\nu, \tau^\nu)\right) = \\ = \lim_{\nu\to\infty} h\left(\{0\}, S(\Delta^\nu, \tau^\nu)\right) \leqslant (y-x)M,$$

which completes the proof in case of $x < y$. For $y < x$ the proof goes with the same patterns. □

4. The Hukuhara derivative and its relation with the Riemann integral

For given $A, B \in clb(X)$, a set $C \in clb(X)$ is called a *difference* $A \overset{*}{-} B$ if $A = B \overset{*}{+} C$ (see [4, p. 210]). The cancellation law implies that if a difference exists, then it is unique. The definition of derivative given below is due to M. Hukuhara (cf. [4, pp. 210–211]). Let $F\colon [a,b] \to clb(X)$ and let $x_0 \in (a,b)$. F is *differentiable* at x_0 if there exist the limits

$$\lim_{x\to x_0^+} \frac{F(x) \overset{*}{-} F(x_0)}{x - x_0} =: F'_+(x_0)$$

and

$$\lim_{x \to x_0^-} \frac{F(x_0) \overset{*}{-} F(x)}{x_0 - x} =: F'_-(x_0)$$

and they are equal. Then a derivative of F at x_0 is defined by $F'(x_0) := F'_+(x_0) = F'_-(x_0)$. In this definition it is assumed that all differences $F(x) \overset{*}{-} F(x_0)$ and $F(x_0) \overset{*}{-} F(x)$ exist for $x > x_0$ and $x < x_0$, respectively, sufficiently close to x_0. Moreover, we say that F is *differentiable* at a if there is $F'_+(a)$ and F is differentiable at b if $F'_-(b)$ exists. In these cases we set $F'(a) := F'_+(a)$ and $F'(b) := F'_-(b)$.

In the sequel we need the following result.

Lemma 12. *If a set-valued function $F\colon [a,b] \to clb(X)$ is continuous, then for each $x \in [a,b)$ the following condition holds:*

$$\frac{1}{\Delta x}\int_x^{x+\Delta x} F(y)dy \to F(x), \quad \textit{for} \quad \Delta x \to 0^+.$$

Proof. From Theorem 9 it follows that

$$h\left(\frac{1}{\Delta x}\int_x^{x+\Delta x} F(y)dy, F(x)\right) = \frac{1}{\Delta x} h\left(\int_x^{x+\Delta x} F(y)dy, \int_x^{x+\Delta x} F(x)dy\right) \leqslant$$
$$\leqslant \frac{1}{\Delta x}\int_x^{x+\Delta x} h\left(F(y), F(x)\right) dy \leqslant \max_{y \in [x, x+\Delta x]} h\left(F(y), F(x)\right).$$

The continuity of F at x completes the proof. □

Now we will show relation between the derivative defined above and the Riemann integral of set-valued function introduced in Section 2.

Theorem 13. *Let $F\colon [a,b] \to clb(X)$ be a continuous set-valued function. Then $G : [a,b] \to clb(X)$ defined by*

$$G(x) := \int_a^x F(y)\, dy$$

is differentiable on $[a,b]$ and moreover, $G' = F$.

Proof. Let us assume that $x \in [a,b)$. Choosing $\Delta x > 0$ so small that $x + \Delta x \leqslant b$, Theorem 8 yields

$$\int_a^{x+\Delta x} F(y)\, dy \;=\; \int_a^x F(y)\, dy \;\overset{*}{+}\; \int_x^{x+\Delta x} F(y)\, dy.$$

This equality implies that

$$G(x+\Delta x) \overset{*}{-} G(x) = \int_a^{x+\Delta x} F(y)\, dy \overset{*}{-} \int_a^{x} F(y)\, dy = \int_x^{x+\Delta x} F(y)\, dy.$$

From Lemma 12 we get

$$h\left(\frac{G(x+\Delta x) \overset{*}{-} G(x)}{\Delta x}, F(x)\right) = h\left(\frac{1}{\Delta x}\int_x^{x+\Delta x} F(y) dy, F(x)\right) \to 0$$

when $\Delta x \to 0$. So $G'_+(x) = F(x)$. The equality $G'_-(x) = F(x)$, $x \in (a,b]$, may be shown in the same way. □

Moreover, we have:

Theorem 14. *Let $F\colon [a,b] \to clb(X)$ be a continuous set-valued function. If $G\colon [a,b] \to clb(X)$ is continuous and satisfies $G'_+(x) = F(x)$ for all $x \in [a,b)$, then the difference $G(b) \overset{*}{-} G(a)$ exists and is equal to*

$$G(b) \overset{*}{-} G(a) = \int_a^b F(x) dx.$$

Proof. Let us consider a function $\varphi\colon [a,b] \to \mathbb{R}$ defined by

$$\varphi(x) = h\left(G(x), G(a) \overset{*}{+} \int_a^x F(y) dy\right), \ x \in [a,b].$$

Clearly, $\varphi(a) = h(G(a), G(a)) = 0$. It suffices to show that $\varphi(b) = 0$. Let us choose any $x \in [a,b)$. We will prove that the right–hand lower Dini's derivative of φ at x, $D_+\varphi(x)$, is nonpositive. Let us take $\Delta x > 0$ such that $x + \Delta x \in (a,b]$. On account of the existence of $G(x+\Delta x) \overset{*}{-} G(x)$ for Δx small enough, properties of the Hausdorff metric h and Theorem 8 we have

$$\begin{aligned}
\varphi(x+\Delta x) &= h\left(G(x+\Delta x), G(a) \overset{*}{+} \int_a^{x+\Delta x} F(y) dy\right) = \\
&= h\left(\left(G(x+\Delta x) \overset{*}{-} G(x)\right) \overset{*}{+} G(x), G(a) \overset{*}{+} \int_a^x F(y) dy \overset{*}{+} \int_x^{x+\Delta x} F(y) dy\right) \leqslant \\
&\leqslant h\left(\left(G(x+\Delta x) \overset{*}{-} G(x)\right) \overset{*}{+} G(x), G(x) \overset{*}{+} \int_x^{x+\Delta x} F(y) dy\right) + \\
&+ h\left(G(x) \overset{*}{+} \int_x^{x+\Delta x} F(y) dy, G(a) \overset{*}{+} \int_a^x F(y) dy \overset{*}{+} \int_x^{x+\Delta x} F(y) dy\right) =
\end{aligned}$$

$$= h\left(G(x+\Delta x) \overset{*}{-} G(x), \int_x^{x+\Delta x} F(y)dy\right) + h\left(G(x), G(a) \overset{*}{+} \int_a^x F(y)dy\right) =$$

$$= h\left(G(x+\Delta x) \overset{*}{-} G(x), \int_x^{x+\Delta x} F(y)dy\right) + \varphi(x).$$

Hence

$$\frac{\varphi(x+\Delta x)-\varphi(x)}{\Delta x} \leqslant \frac{1}{\Delta x}\, h\left(G(x+\Delta x) \overset{*}{-} G(x), \int_x^{x+\Delta x} F(y)dy\right) \leqslant$$

$$\leqslant h\left(\frac{G(x+\Delta x) \overset{*}{-} G(x)}{\Delta x}, F(x)\right) + h\left(F(x), \frac{1}{\Delta x}\int_x^{x+\Delta x} F(y)dy\right).$$

Since the right-hand side of the above inequality tends to zero when $\Delta x \to 0^+$ (compare with Lemma 12),

$$D_+\varphi(x) = \liminf_{\Delta x \to 0^+} \frac{\varphi(x+\Delta x)-\varphi(x)}{\Delta x} \leqslant 0.$$

So on account of the Zygmund's Lemma (see. [5, Corollary, p. 183]) the function φ is decreasing. At the same time φ takes only nonnegative values and $\varphi(a) = 0$. Finally, $\varphi(x) = 0$ for all $x \in [a, b)$ and from the continuity of φ we have

$$\varphi(b) = 0.$$

□

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Omówienie

W pracy przedstawiono definicję całki Riemanna z funkcji wielowartościowej o wartościach będących niepustymi domkniętymi ograniczonymi i wypukłymi podzbiorami rzeczywistej przestrzeni Banacha. Jest ona uogólnieniem definicji wprowadzonej przez Dinghasa i Hukuhary dla przypadku zwartych i wypukłych podzbiorów przestrzeni $\mathbb{R}^n$. Dokładną definicję zawiera rozdział 2. W kolejnym rozdziale rozważamy podstawowe własności tej całki oraz porównujemy jej zachowanie z przypadkiem całki Riemanna z funkcji rzeczywistej, podając między innymi kontrprzykłady w sytuacjach, gdy występują istotne różnice. Ostatni rozdział przedstawia związek całki Riemanna z funkcji wielowartościwych z pochodną Hukuhary.