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APPROXIMATED SNELL ENVELOPE AND ITS APPLICATIONS

Abstract. This paper proposes some mild conditions on underlying stochastic process of optimal stopping and some approximations are proposed for Snell envelope techniques. The aim is to simplify the computation of conditional expectations which are necessary in obtaining the sequential backward Snell auxiliary process. Then, by applying these approximations to return process of a financial asset, the behaviors of optimal stopping times at which the expectation of return process is optimized are studied. Here, it is assumed that the mean corrected return process is of GARCH type.

1. Introduction

In optimal control literature, optimal stopping concerns with the problem of choosing a time to take a given action based on sequential observations. Optimal stopping has many applications in statistics (in sequential hypothesis testing), mathematics (in optimal search), operation research (in secretary problem), economics (portfolio management), trading (in finding momentum positions) and finance (in pricing American option), see Marti (2004). In most of cases, the optimal stopping problem is written as Bellman equation and it is solved using

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dynamic programming and backward induction. This method has many advantages, but it is too time-consuming. Generally, there are three methods to solve optimal stopping problems, including the Markovian approach, the martingale approach, and the Snell envelope technique. The Snell method is a frequently used method in mathematical finance, see Wang (2010). Often, this technique is based on constructing conditional expectations and maximizing them and the underlying process. However, from a computational point of view, except for independent increments such as Brownian motion and random walk, the closed-form of conditional expectation does not exist. Although Longstaff and Schwartz (2001) proposed a simple regression method for American option pricing problems, the Snell technique is generally difficult to apply, see Peskir and Shiryaev (2006).

Suppose that X_t, $t = 1, 2, \dots$ is a discrete time stochastic process and $\mathcal{F}_t$ is the natural filtration (σ-field, information set) made by $(X_1, \dots, X_t)$. The optimal stopping problem is to find a stopping time τ which maximizes $E(X_t)$. For finite horizon case, when $t = 1, \dots, n$, the Snell envelope technique (see Ferguson, 2007) defines an auxiliary process g_t, $1 \le t \le n$, as follows:

$$g_t = \begin{cases} \max\left(X_t, E(g_{t+1}|\mathcal{F}_t)\right), & 1 \le t \le n-1, \\ X_n, \quad t = n. \end{cases}$$

Such an approach and idea in the analysis of stochastic processes is being used in algorithmic trading. To this end, let r_t be the t-th return of a specific stock and let μ_t and σ_t be the mean and the standard deviation of the return process, respectively. For simplicity of arguments, assume that

$$r_t = \mu_t + \sigma_t z_t,$$

where z_t is a white noise sequence with a specific density function ϕ and the corresponding distribution and survive functions Φ and $\overline{\Phi}$, respectively. Let $X_t = r_t - \mu_t$. Here, σ_t^2 obeys a GARCH series defined by

$$\sigma_t^2 = w + \alpha\sigma_{t-1}^2 + \beta X_{t-1}^2.$$

We are interested in finding the first time (stopping time) τ at which X_t reaches its maximum, i.e., r_t is the farthest from the mean μ_t.

The Snell envelope technique is based on a backward induction method that uses a dynamic programming approach. As stated, proposing analytical solutions for Snell's method requires closed-forms of recursive conditional expectations, which is hard in most cases. The asymptotic solutions provide a suitable alternative approach. In the case of independent observations, Lustri et al. (2020) proposed a recursive relation for expectations of Snell's sequences. In the present paper, under some mild conditions, similar recursive relations are proposed in the case of dependent observations, as well as its application in trading is proposed. The simulation results are also described.

2. Formulation

The following lemma will be used in the proposed extension of Snell envelope.

Lemma 1. *Suppose that X is a continuous random variable with density function f, Y is another continuous random variable and a is a constant real number. Then*

(i) $E(\max(X-a,0)) = \int_a^\infty P(X>z)dz$.

(ii) $E(\max(X-Y,0)) = E\int_Y^\infty P(X>z|Y)dz$.

(iii) If $E|Y-E(Y)| \to 0$, *then* $E(\max(X-Y,0)) = \int_{E(Y)}^\infty P(X>z)dz$.

Proof. To show (i), we can write

$$\begin{aligned} E(\max(X-a,0)) &= \int_{-\infty}^{\infty} \max(x-a,0) f(x)dx = \\ &= \int_a^\infty (x-a)f(x)dx = \int_a^\infty \int_a^x dz f(x)dx = \\ &= \int_a^\infty \int_z^\infty f(x)dxdz = \int_a^\infty P(X>z)dz. \end{aligned}$$

To show (ii), we can write

$$E(\max(X-Y,0)) = E(E(\max(X-Y,0))|Y) = E\int_Y^\infty P(X>z|Y)dz.$$

For (iii), observe that

$$E\left|\int_Y^\infty P(X>z|Y)dz - \int_{E(Y)}^\infty P(X>z|Y)dz\right| \le E|Y-E(Y)| \to 0.$$

Therefore

$$E\int_{Y}^{\infty} P(X>z|Y)dz = E\int_{E(Y)}^{\infty} P(X>z|Y)dz =$$
$$= \int_{E(Y)}^{\infty} E\left(P(X>z|Y)\right)dz = \int_{E(Y)}^{\infty} P(X>z)dz.$$

□

Remark 2. In Lemma 1 and its proof, it is assumed that $E|Y - E(Y)| \to 0$. To clarify the type of this convergence, suppose that $Y = Y_\vartheta$ is indexed by some deterministic parameter $\vartheta > 0$ and as ϑ gets large or small, then $E|Y_\vartheta - E(Y_\vartheta)|$ goes to zero as a function of ϑ. As it was seen in the proof, it suffices that $\mathrm{var}(Y_\vartheta)$ tends to zero. As an example, let $\vartheta = n$ and let $Y_\vartheta = \overline{W}_n$ be the average of n copies of finite variances random variables W_i, $i = 1, \ldots, n$.

2.1. Asymptotic envelope

To propose extension, let us define:

$$Y_t := E(g_{t+1}|\mathcal{F}_t), \quad v_t := E(g_t).$$

Then $g_t = Y_t + \max(X_t - Y_t, 0)$. Assume that $E|g_t - v_t|$ is close to zero. Then g_t is approximated well by its expectation v_t. Thus

$$v_t = v_{t+1} + E(\max(X_t - Y_t, 0)).$$

Here and in Lemma 3, the condition for $E\,|g_t - v_t|$ to be small is sought. Note that

$$E\,|g_t - v_t| = E\sqrt{(g_t - v_t)^2} \le \sqrt{E(g_t - v_t)^2} = \sqrt{\mathrm{var}(g_t)}.$$

The inequality in the middle is satisfied by Jensen's inequality and the fact that the function $\sqrt{x}$ is concave. So, it is enough to find a condition for $\mathrm{var}(g_t)$ to be small. Let us define

$$M_n := \max(\mathrm{var}(X_1), \ldots, \mathrm{var}(X_n)).$$

The required condition can be stated as follows.

Lemma 3. *We have* $\mathrm{var}(g_t) \leq M_n$ *for* $t = 1, \dots, n$. *In particular, if* $M_n \to 0$, *then all* $\mathrm{var}(g_t)$ *are close to zero.*

Proof. Since $g_n = X_n$, we get $\mathrm{var}\,(g_n) \leq \mathrm{var}\,(X_n) \leq M_n$. Next, we have $g_{n-1} = \max\,(X_{n-1}, E(X_n|\mathcal{F}_{n-1}))$, which equals to X_{n-1} in the case

$$\mathrm{var}\,(g_{n-1}) \leq \mathrm{var}(X_{n-1}) \leq M_n,$$

or to $E(X_n|\mathcal{F}_{n-1})$ in the case

$$\mathrm{var}(E(X_n|\mathcal{F}_{n-1})) \leq \mathrm{var}(X_n) \leq M_n.$$

Using backward induction, it is concluded that $\mathrm{var}(g_t) \leq M_n$. This completes the proof. □

Note that $\mathrm{var}(Y_t) \leq \mathrm{var}(g_t) \leq M_n$. Therefore $E|Y_t - E\,(Y_t)\,| \to 0$ as $M_n \to 0$. Using Lemma 1, it is concluded that

$$E\,(\max\,(X_t - Y_t, 0)\ ) = \int_{v_{t+1}}^{\infty} P\,(X_{t+1} > z)\,dz.$$

Therefore, it is concluded that

$$v_t = v_{t+1} + \int_{v_{t+1}}^{\infty} P\,(X_{t+1} > z)\,dz.$$

Let $t_0 = 0$ and t_n $(n \geq 1)$ be positive real indices such that $t_n - t_{n-1} = h$ for some predetermined positive h, and assume that $v_{t_n} = v_n$. Following Lustri et al. (2020), it can be seen that the asymptotic behaviors of $-v_n'$ and $\int_{v_n}^{\infty} P\,(X_n > z)\,dz$ are the same. Similarly, the asymptotic behaviors of $\frac{v_n''}{v_n'}$ and $h_n(v_n) := P(X_n > v_n)$ are the same. Thus, we can write

$$\begin{aligned} -v_n' &\sim \int_{v_n}^{\infty} P\,(X_n > z)\,dz, \\ \frac{v_n''}{v_n'} &\sim h_n\,(v_n)\,. \end{aligned}$$

Following Suli and Mayers (2003), the forward numerical approximation for v_n' is given by

$$\frac{v_{n+1} - v_n}{h},$$

and the suitable approximation for v''_n is given by

$$\frac{1}{h}\left(\frac{v_{n+2}-v_{n+1}}{h}-\frac{v_{n+1}-v_n}{h}\right).$$

Therefore, both $\frac{v''_n}{v'_n}$ and $h_n(v_n)$ can be numerically approximated by

$$\frac{v_{n+2}-2v_{n+1}+v_n}{h(v_{n+1}-v_n)}.$$

This justifies the following proposition.

Proposition 4. *Let $v_t := E(g_t)$ and $h_n(z) := P(X_n > z)$. If $M_n \to 0$, then*

(a) $v_t = v_{t+1} + \int_{v_t}^{\infty} h_n(z)\, dz$,

(b) $v'_n \sim -\int_{v_n}^{\infty} h_n(z)\, dz$,

(c) $h_n(v_n) \sim \frac{v''_n}{v'_n}$, $h_n(v_n) \sim \frac{v_{n+2}-2v_{n+1}+v_n}{h(v_{n+1}-v_n)}$.

In the case of GARCH series, we have

$$\operatorname{var}(X_n) = E\left(X_t^2\right) = E\left(\sigma_t^2\right) = \frac{w}{1-(\alpha+\beta)}.$$

Therefore, it is enough to ensure that $\frac{w}{1-(\alpha+\beta)}$ is small. We also have

$$h_n(x) = P(\sigma_n z_n > x) = E\left(\overline{\Phi}\left(\frac{x}{\sqrt{\sigma_n^2}}\right)\right).$$

To approximate the last expectation, the following lemma will be used.

Lemma 5. *Let γ and δ be, respectively, the long-term mean and the standard deviation of σ_t^2. Then for every twice differentiable measurable function $\zeta(\cdot)$, we have*

$$E\left(\zeta\left(\sigma_n^2\right)\right) \approx \zeta(\gamma) + \frac{1}{2}\zeta''(\gamma)\delta^2.$$

The proof of Lemma 5 is based on the Taylor expansion of function of a random variable and taking expectation. It is a routine work in the field of large sample theory (see Lehmann, 2018). Using Lemma 5, we get

$$E\left(\overline{\Phi}\left(\frac{x}{\sqrt{\sigma_n^2}}\right)\right) \approx \overline{\Phi}\left(\frac{x}{\sqrt{\gamma}}\right) - \frac{\delta^2\gamma^{-3}x}{8}\left(3\sqrt{\gamma}\phi\left(\frac{x}{\sqrt{\gamma}}\right) + x\phi'\left(\frac{x}{\sqrt{\gamma}}\right)\right).$$

In consequence, we obtain the following proposition.

Proposition 6. *The function $h_n(x)$ from Proposition 4 can be approximated as follows*

$$h_n(x) \approx \overline{\Phi}\left(\frac{x}{\sqrt{\gamma}}\right) - \frac{\delta^2\gamma^{-3}x}{8}\left(3\sqrt{\gamma}\phi\left(\frac{x}{\sqrt{\gamma}}\right) + x\phi'\left(\frac{x}{\sqrt{\gamma}}\right)\right),$$

where the unconditional mean γ and the standard deviation δ of GARCH series satisfy

$$\gamma = \frac{w}{1-(\alpha+\beta)}, \qquad \delta^2 = \frac{\beta^2 \mathrm{var}\left(z_t^2\right) E^2\left(\sigma_t^2\right)}{1-\alpha^2-\beta^2 E\left(z_t^4\right)}.$$

If z_t has the standard normal distribution, then $\mathrm{var}\left(z_t^2\right) = 2$ *and* $E\left(z_t^4\right) = 3$.

2.2. Simulations

In this section, a simple method for approximation of v_t is proposed, which allows to use the Monte Carlo simulations. To this end, first suppose that σ_t^2 is a non-random real function and remember that

$$v_t - v_{t+1} = \int_{v_{t+1}}^{\infty} P\left(\sigma_{t+1} z_{t+1} > z\right) dz.$$

Assume that $\frac{v_{t+1}}{\sigma_{t+1}} = \lambda$ is constant for each t. Then

$$\lambda \frac{\sigma_t - \sigma_{t+1}}{\sigma_{t+1}} = \int_{\lambda}^{\infty} P\left(z_{t+1} > z\right) dz.$$

By differentiation with respect to λ, it can be seen that $P\left(z_{t+1} \leq \lambda\right) = \frac{\sigma_t}{\sigma_{t+1}}$. Thus λ is the $\frac{\sigma_t}{\sigma_{t+1}}$-th quantile of z_{t+1}. Again, considering σ_t as a GARCH series and simulating it by Monte Carlo method, the empirical distribution of z_{t+1} is fitted. Furthermore, the empirical value of $\frac{\sigma_t}{\sigma_{t+1}}$ is forecasted. Therefore λ is estimated by the $\frac{\sigma_t}{\sigma_{t+1}}$-th quantile of distribution of z_{t+1}.

Consider the daily log-return of *Apple Inc.* for period of 21 July 2021 to 20 June 2022, including 252 observations. The historical prices s_t are taken from https://www.nasdaq.com, and the daily log–returns are computed using $r_t = \log(s_t) - \log(s_{t-1})$. Here, $\mu_t = -0.00068$ and $w = 4.22 \times 10^{-5}$, $\alpha = 0.861$, $\beta = 0.034$. Assuming z_t as a sequence of independent and standard normally distributed random variables, Fig. 1 gives the time series plot of v_t. Here, the skew and kurtosis of z_t are -0.021 and 2.96, respectively, which shows that z_t is normally distributed.

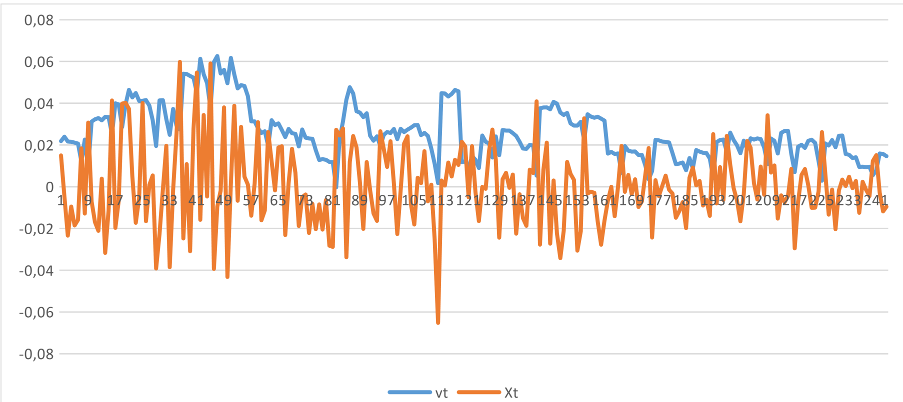

Fig. 1. Mean corrected series X_t vs v_t

Remark 7. The assumption of normality for z_t does not hold in practice. Here, the Monte Carlo simulation may be applied to derive the empirical distribution of z_t. In the case of heavy-tailed distributions, suitable ones such as t-student distribution is reasonable. In the case of thick-tail distributions, usually, it is reasonable to decompose z_t as $\text{sign}(z_t) \times |z_t|$ and to use a suitable copula function, which is based on empirical covariance structure between the sign and absolute value of z_t with marginal distributions as Bernoulli law for $0.5(1 + \text{sign}(z_t))$ and the normal distribution for a Box-Cox transformation of $|z_t|$. The decomposition of normal random variables to their signs and absolute values are also interesting. An application of this analysis are the stock or foreign exchange markets predictions, where investors are interested in the sign of change indicating an increase or decrease in price before predicting the amount of the change in order to take an appropriate trading position. Also, it should be mentioned that for the Monte Carlo simulation of the product of two random variables, it is necessary to choose appropriate copula function. For comprehensive review in these fields, see Mai and Scherer (2012).

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