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COMPLEX OSCILLATION OF A SECOND ORDER LINEAR DIFFERENTIAL EQUATION WITH ENTIRE COEFFICIENTS OF (α, β)-ORDER

Abstract. In this paper we study distribution of zeros and growth of solutions of second order linear equations depending on the coefficients of the equation and their (α, β)-order. We obtain results in general form, which considerably extend some results from [21].

1. Introduction, Definitions and Notations

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of entire and meromorphic functions which are available in [11,18,20,26–28] and therefore we do not explain those in details. It is well-known that the theory of complex linear differential equations has been developed since 1960s. Several

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authors have investigated the second order linear differential equation

$$f'' + A(z)f = 0, \tag{1}$$

when $A(z)$ is an entire function or a meromorphic function of finite order or finite iterated order, and have obtained many results about the interaction between the solutions and the coefficient of (1) (see [1–3, 17]). Moreover, some authors have investigated the exponent of convergence of zero sequence and pole sequence of the solutions of second order differential equations and have obtained some interesting results (see [6, 7, 17, 25]).

We denote the linear measure and the logarithmic measure of a set $E \subset (1, +\infty)$ by $mE = \int_E dx$ and $m_l E = \int_E \frac{dx}{x}$. Now let L be a class of continuous functions α, non-negative on $(-\infty, +\infty)$, such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ and $\alpha(x) \to +\infty$ as $x_0 \leq x \to +\infty$.

During the past decades, several authors made close investigations on the properties of entire functions related to (α, β)-order in some different direction. Recently Mulyava et al. [19] have investigated the properties of solutions of a heterogeneous differential equation of the second order under some different conditions and have obtained several interesting results. For details one may see [19]. Now it is interesting to investigate distribution of zeros and growth of solutions of second order linear equations depending on the coefficients of the equation and their (α, β)-order, which is the main aim of this paper. For this purpose, we rewrite the definition of the (α, β)-order of a meromorphic function in the following way after giving a minor modification to the original definition (e.g. see, [19, 22]):

Definition 1. *Let $\alpha \in L$ and $\beta \in L$. The (α, β)-order denoted by $\sigma_{(\alpha,\beta)}[f]$ and (α, β)-lower order denoted by $\mu_{(\alpha,\beta)}[f]$ of a meromorphic function f are, respectively, defined by*

$$\sigma_{(\alpha,\beta)}[f] = \limsup_{r \to +\infty} \frac{\alpha(\log T(r, f))}{\beta(\log r)} \quad \textit{and} \quad \mu_{(\alpha,\beta)}[f] = \liminf_{r \to +\infty} \frac{\alpha(\log T(r, f))}{\beta(\log r)},$$

where $T(r, f)$ is the Nevanlinna characteristic function of f.

Example 2. Let f be a meromorphic function. One can see that $\alpha(r) = \log^{[p]} r$, $(p \geq 0)$ and $\beta(r) = \log^{[q]} r$, $(q \geq 0)$ belong to the class L, where $\log^{[k]} x = \log(\log^{[k-1]} x)$ $(k \geq 1)$, with convention that $\log^{[0]} x = x$. So, when $p = 0$ and $q = 0$, i.e., $\alpha(r) = \beta(r) = r$, the Definition 1 coincides with the usual order and lower order, when $\alpha(r) = \log^{[p-1]} r$ $(p \geq 1)$ and $\beta(r) = r$, we obtain the iterated $p-$order and the iterated lower $p-$order (see [17], [23]), moreover when $\alpha(r) = \log^{[p-1]} r$ and $\beta(r) = \log^{[q-1]} r$, $(p \geq q \geq 1)$, we get the (p,q)-order and the lower (p,q)-order (see [14], [15]). Finally, if $\alpha(r) = \varphi(e^r)$, where φ is an increasing unbounded function on $[1,+\infty)$ and $\beta(r) = r$, we obtain the $\varphi-$order and the lower $\varphi-$order (see [4], [8]).

Let f be a meromorphic function, $n(r,f)$ be the number of poles of $f(z)$ in $|z| \leq r$, each counted with correct multiplicity, and let $\overline{n}(r,f)$ be the number of poles, where each multiple pole is counted only once. Similarly to Definition 1 we can also define the (α,β)-exponent of convergence of the zero sequence and (α,β)-exponent of convergence of the distinct zero sequence of a meromorphic function f in the following way:

Definition 3. *Let $\alpha \in L$ and $\beta \in L$. The (α,β)-exponent of convergence of the zero sequence of a meromorphic function f, denoted by $\lambda_{(\alpha,\beta)}[f]$, is defined by*

$$\lambda_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log n(r,1/f))}{\beta(\log r)}.$$

Similarly, the (α,β)-exponent of convergence of the distinct zero sequence of f, denoted by $\overline{\lambda}_{(\alpha,\beta)}[f]$, is defined by

$$\overline{\lambda}_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log \overline{n}(r,1/f))}{\beta(\log r)}.$$

We say that $\alpha \in L_1$, if $\alpha \in L$ and $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$ and $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1+o(1))\alpha(x)$ as $x \to +\infty$ for each fixed $c \in (0,+\infty)$. It is clear that $L_{si} \subset L_1$. Now we add two conditions on α and β:

(i) α and β always denote the functions belonging to L_{si} and L_1, respectively, and

(ii) $\alpha(\log x) = o(\beta(x))$ as $x \to +\infty$.

Throughout this paper, we assume that α and β always satisfy the above two conditions unless otherwise specifically stated.

Proposition 4. *Let f_1, f_2 be non-constant meromorphic functions with $\sigma_{(\alpha,\beta)}[f_1]$ and $\sigma_{(\alpha,\beta)}[f_2]$ as their (α,β)-order. Then*

(i) $\sigma_{(\alpha,\beta)}[f_1 \pm f_2] \leq \max\{\sigma_{(\alpha,\beta)}[f_1],\, \sigma_{(\alpha,\beta)}[f_2]\}$,

(ii) $\sigma_{(\alpha,\beta)}[f_1 \cdot f_2] \leq \max\{\sigma_{(\alpha,\beta)}[f_1], \sigma_{(\alpha,\beta)}[f_2]\}$,

(iii) *if* $\sigma_{(\alpha,\beta)}[f_1] \neq \sigma_{(\alpha,\beta)}[f_2]$, *then* $\sigma_{(\alpha,\beta)}[f_1 \pm f_2] = \max\{\sigma_{(\alpha,\beta)}[f_1], \sigma_{(\alpha,\beta)}[f_2]\}$,

(iv) *if* $\sigma_{(\alpha,\beta)}[f_1] \neq \sigma_{(\alpha,\beta)}[f_2]$, *then* $\sigma_{(\alpha,\beta)}[f_1 \cdot f_2] = \max\{\sigma_{(\alpha,\beta)}[f_1], \sigma_{(\alpha,\beta)}[f_2]\}$.

Proof. (i) Without loss of generality, we assume that $\sigma_{(\alpha,\beta)}[f_1] \leq \sigma_{(\alpha,\beta)}[f_2] < +\infty$. From the definition of (α, β)-order, for any $\varepsilon > 0$, we obtain for all sufficiently large values of r that

$$T(r, f_1) < \exp(\alpha^{-1}((\sigma_{(\alpha,\beta)}[f_1] + \varepsilon)\beta(\log r))) \tag{2}$$

and

$$T(r, f_2) < \exp(\alpha^{-1}((\sigma_{(\alpha,\beta)}[f_2] + \varepsilon)\beta(\log r))). \tag{3}$$

Since $T(r, f_1 \pm f_2) \leq T(r, f_1) + T(r, f_2) + \log 2$ for all large r, we get from (2) and (3), for all sufficiently large values of r, that

$$\begin{aligned} T(r, f_1 \pm f_2) &< 2\exp(\alpha^{-1}((\sigma_{(\alpha,\beta)}[f_2] + \varepsilon)\beta(\log r))) + \log 2, \\ \textit{i.e.},\ T(r, f_1 \pm f_2) &< 3\exp(\alpha^{-1}((\sigma_{(\alpha,\beta)}[f_2] + \varepsilon)\beta(\log r))), \\ \textit{i.e.},\ \frac{1}{3}T(r, f_1 \pm f_2) &< \exp(\alpha^{-1}((\sigma_{(\alpha,\beta)}[f_2] + \varepsilon)\beta(\log r))), \\ \textit{i.e.},\ \log T(r, f_1 \pm f_2) - \log 3 &< \alpha^{-1}((\sigma_{(\alpha,\beta)}[f_2] + \varepsilon)\beta(\log r)). \end{aligned}$$

We can write

$$\log T(r, f_1 \pm f_2) - \log 3 = \left(1 - \frac{\log 3}{\log T(r, f_1 \pm f_2)}\right) \log T(r, f_1 \pm f_2).$$

Since $\frac{\log 3}{\log T(r, f_1 \pm f_2)} \to 0$ as $r \to +\infty$ and $\alpha \in L_1$, we obtain

$$(1 + o(1))\alpha\left(\log T(r, f_1 \pm f_2)\right) = \alpha((1 - \frac{\log 3}{\log T(r, f_1 \pm f_2)}) \log T(r, f_1 \pm f_2))$$

$$\leq \left(\sigma_{(\alpha,\beta)}[f_2] + \varepsilon\right) \beta\left(\log r\right),$$

which implies that

$$\limsup_{r \to +\infty} \frac{(1 + o(1))\alpha(\log T(r, f_1 \pm f_2))}{\beta(\log r)} \leq \sigma_{(\alpha,\beta)}[f_2] + \varepsilon$$

holds for any $\varepsilon > 0$. Hence

$$\sigma_{(\alpha,\beta)}[f_1 \pm f_2] \leq \max\{\sigma_{(\alpha,\beta)}[f_1], \sigma_{(\alpha,\beta)}[f_2]\}. \tag{4}$$

(iii) Further, without loss of any generality, let $\sigma_{(\alpha,\beta)}[f_1] < \sigma_{(\alpha,\beta)}[f_2] < +\infty$ and $f = f_1 \pm f_2$. Then in view of (4) we get that $\sigma_{(\alpha,\beta)}[f] \leq \sigma_{(\alpha,\beta)}[f_2]$. As $f_2 = \pm(f - f_1)$, in this case we obtain that $\sigma_{(\alpha,\beta)}[f_2] \leq \max\{\sigma_{(\alpha,\beta)}[f], \sigma_{(\alpha,\beta)}[f_1]\}$. As we assume that $\sigma_{(\alpha,\beta)}[f_1] < \sigma_{(\alpha,\beta)}[f_2]$, therefore we have $\sigma_{(\alpha,\beta)}[f_2] \leq \sigma_{(\alpha,\beta)}[f]$ and hence $\sigma_{(\alpha,\beta)}[f] = \sigma_{(\alpha,\beta)}[f_2] = \max\{\sigma_{(\alpha,\beta)}[f_1], \sigma_{(\alpha,\beta)}[f_2]\}$.

(ii) and (iv) Similarly, from $T(r, f_1 \cdot f_2) \leq T(r, f_1) + T(r, f_2)$ for all large r, we can also get

$$\sigma_{(\alpha,\beta)}[f_1 \cdot f_2] \leq \max\{\sigma_{(\alpha,\beta)}[f_1],\ \sigma_{(\alpha,\beta)}[f_2]\}$$

and if $\sigma_{(\alpha,\beta)}[f_1] \neq \sigma_{(\alpha,\beta)}[f_2]$, then

$$\sigma_{(\alpha,\beta)}[f_1 \cdot f_2] = \max\{\sigma_{(\alpha,\beta)}[f_1],\ \sigma_{(\alpha,\beta)}[f_2]\},$$

which completes the proof of Proposition 4. □

Proposition 5. *Let f_1 and f_2 be non-constant meromorphic functions with $\sigma_{(\alpha(\log),\beta)}[f_1]$ and $\sigma_{(\alpha(\log),\beta)}[f_2]$ as their $(\alpha(\log),\beta)$-order. Then*

(i) $\sigma_{(\alpha(\log),\beta)}[f_1 \pm f_2] \leq \max\{\sigma_{(\alpha(\log),\beta)}[f_1],\ \sigma_{(\alpha(\log),\beta)}[f_2]\}$,

(ii) $\sigma_{(\alpha(\log),\beta)}[f_1 \cdot f_2] \leq \max\{\sigma_{(\alpha(\log),\beta)}[f_1],\ \sigma_{(\alpha(\log),\beta)}[f_2]\}$,

(iii) if $\sigma_{(\alpha(\log),\beta)}[f_1] \neq \sigma_{(\alpha(\log),\beta)}[f_2]$, *then*

$$\sigma_{(\alpha(\log),\beta)}[f_1 \pm f_2] = \max\{\sigma_{(\alpha(\log),\beta)}[f_1], \sigma_{(\alpha(\log),\beta)}[f_2]\},$$

(iv) if $\sigma_{(\alpha(\log),\beta)}[f_1] \neq \sigma_{(\alpha(\log),\beta)}[f_2]$, *then*

$$\sigma_{(\alpha(\log),\beta)}[f_1 \cdot f_2] = \max\{\sigma_{(\alpha(\log),\beta)}[f_1], \sigma_{(\alpha(\log),\beta)}[f_2]\}.$$

Since $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$, the proof of Proposition 5 would run parallelly to that of Proposition 4. We omit the details.

Proposition 6. *(i) If f is an entire function, then*

$$\sigma_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log T(r, f))}{\beta(\log r)} = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} M(r, f))}{\beta(\log r)}$$

and

$$\mu_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\alpha(\log T(r,f))}{\beta(\log r)} = \liminf_{r\to+\infty} \frac{\alpha(\log^{[2]} M(r,f))}{\beta(\log r)},$$

where $M(r,f) = \max\{|f(z)|\colon |z| = r\}$.

(ii) If f *is a meromorphic function, then*

$$\lambda_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log n(r,1/f))}{\beta(\log r)} = \limsup_{r\to+\infty} \frac{\alpha(\log N(r,1/f))}{\beta(\log r)}$$

and

$$\overline{\lambda}_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log \overline{n}(r,1/f))}{\beta(\log r)} = \limsup_{r\to+\infty} \frac{\alpha(\log \overline{N}(r,1/f))}{\beta(\log r)},$$

where $N(r,1/f)$ *and* $\overline{N}(r,1/f)$ *are the corresponding counting functions of poles of* $1/f$.

Proof. (i) By the inequality $T(r,f) \le \log^+ M(r,f) \le \frac{R+r}{R-r} T(R,f)$ $(0 < r < R)$ (cf. [11]) for an entire function f, set $R = \eta r$ $(\eta > 1)$, we have

$$T(r,f) \le \log^+ M(r,f) \le \frac{\eta+1}{\eta-1} T(\eta r, f). \tag{5}$$

By (5), $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$ and $\beta((1+o(1))x) = (1+o(1))\beta(x)$ as $x \to +\infty$, it is easy to see that conclusion (i) holds.

(ii) Without loss of generality, assume that $f(0) \ne 0$, then $N(r,1/f) = \int_0^r \frac{n(t,1/f)}{t} dt$. We have

$$N(r,1/f) - N(r_0,1/f) = \int_{r_0}^{r} \frac{n(t,1/f)}{t} dt \le n(r,1/f) \log \frac{r}{r_0} \ (0 < r_0 < r),$$

that is

$$N(r,1/f) \le N(r_0,1/f) + n(r,1/f) \log \frac{r}{r_0} \ (0 < r_0 < r),$$

$$i.e.,\ N(r,1/f) \le \left(1 + \frac{N(r_0,1/f)}{n(r,1/f)\log \frac{r}{r_0}}\right) n(r,1/f) \log \frac{r}{r_0} \ (0 < r_0 < r),$$

which implies

$$\log N(r,1/f) \le \log n(r,1/f) + \log\log r$$
$$+ \log\left(1 - \frac{\log r_0}{\log r}\right) + \log\left(1 + \frac{N(r_0,1/f)}{n(r,1/f)\log \frac{r}{r_0}}\right) \ (0 < r_0 < r), \tag{6}$$

then by (6), we have

$$\begin{aligned}
\limsup_{r\to+\infty}\frac{\alpha(\log N(r,1/f))}{\beta(\log r)} &\le \limsup_{r\to+\infty}\frac{\alpha\left((1+o\,(1))\left(\log n(r,1/f)+\log^{[2]} r\right)\right)}{\beta(\log r)}\\
&\le \limsup_{r\to+\infty}\frac{(1+o\,(1))\,\alpha(\log n(r,1/f)+\log^{[2]} r)}{\beta(\log r)}\\
&\le \limsup_{r\to+\infty}\frac{\alpha(2\max\{\log n(r,1/f),\ \log^{[2]} r\})}{\beta(\log r)}\\
&= \limsup_{r\to+\infty}\frac{(1+o\,(1))\max\{\alpha(\log n(r,1/f)),\ \alpha(\log^{[2]} r)\}}{\beta(\log r)}\\
&= \limsup_{r\to+\infty}\frac{\alpha(\log n(r,1/f))+\alpha(\log^{[2]} r)}{\beta(\log r)}\\
&\le \limsup_{r\to+\infty}\frac{\alpha(\log n(r,1/f))}{\beta(\log r)}+\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]} r)}{\beta(\log r)}\\
&= \limsup_{r\to+\infty}\frac{\alpha(\log n(r,1/f))}{\beta(\log r)}, \qquad (7)
\end{aligned}$$

since $\alpha(\log x)=o(\beta(x))$ as $x\to+\infty$ we have $\frac{\alpha(\log^{[2]} r)}{\beta(\log r)}\to 0$ as $r\to+\infty$.

On the other hand, we have

$$\begin{aligned}
N(er,1/f) &= \int_0^{er}\frac{n(t,1/f)}{t}dt \ge \int_r^{er}\frac{n(t,1/f)}{t}dt\\
&\ge n(r,1/f)\log e = n(r,1/f). \qquad (8)
\end{aligned}$$

By (8) and the condition $\beta((1+o(1))x)=(1+o(1))\beta(x)$ as $x\to+\infty$, we have

$$\limsup_{r\to+\infty}\frac{\alpha(\log N(er,1/f))}{\beta(\log r)} \ge \limsup_{r\to+\infty}\frac{\alpha(\log n(r,1/f))}{\beta(\log r)}.$$

We can write

$$\begin{aligned}
\limsup_{r\to+\infty}\frac{\alpha(\log N(er,1/f))}{\beta(\log r)} &= \limsup_{r\to+\infty}\frac{\alpha(\log N(er,1/f))}{\beta(\log er-\log e)}\\
&= \limsup_{r\to+\infty}\frac{\alpha(\log N(er,1/f))}{\beta(\left(1-\frac{1}{\log er}\right)\log er)}\\
&= \limsup_{r\to+\infty}\frac{\alpha(\log N(er,1/f))}{\beta((1+o(1))\log er)}\\
&= \limsup_{r\to+\infty}\frac{\alpha(\log N(er,1/f))}{(1+o(1))\,\beta(\log er)}\\
&= \limsup_{r\to+\infty}\frac{\alpha(\log N(r,1/f))}{\beta(\log r)},
\end{aligned}$$

it follows that

$$\limsup_{r\to+\infty}\frac{\alpha(\log N(r,1/f))}{\beta(\log r)} \geq \limsup_{r\to+\infty}\frac{\alpha(\log n(r,1/f))}{\beta(\log r)}. \tag{9}$$

By (7) and (9), it is easy to see that

$$\lambda_{(\alpha,\beta)}[f]=\limsup_{r\to+\infty}\frac{\alpha(\log n(r,1/f))}{\beta(\log r)}=\limsup_{r\to+\infty}\frac{\alpha(\log N(r,1/f))}{\beta(\log r)}.$$

By the same proof as above, we can obtain the conclusion

$$\overline{\lambda}_{(\alpha,\beta)}[f]=\limsup_{r\to+\infty}\frac{\alpha(\log \overline{n}(r,1/f))}{\beta(\log r)}=\limsup_{r\to+\infty}\frac{\alpha(\log \overline{N}(r,1/f))}{\beta(\log r)}.$$

□

Proposition 7. *(i) If f is an entire function, then*

$$\sigma_{(\alpha(\log),\beta)}[f]=\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}T(r,f))}{\beta(\log r)}=\limsup_{r\to+\infty}\frac{\alpha(\log^{[3]}M(r,f))}{\beta(\log r)}$$

and

$$\mu_{(\alpha(\log),\beta)}[f]=\liminf_{r\to+\infty}\frac{\alpha(\log^{[2]}T(r,f))}{\beta(\log r)}=\liminf_{r\to+\infty}\frac{\alpha(\log^{[3]}M(r,f))}{\beta(\log r)}.$$

(ii) If f is a meromorphic function, then

$$\lambda_{(\alpha(\log),\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} n(r,1/f))}{\beta(\log r)} = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} N(r,1/f))}{\beta(\log r)}$$

and

$$\overline{\lambda}_{(\alpha(\log),\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} \overline{n}(r,1/f))}{\beta(\log r)} = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} \overline{N}(r,1/f))}{\beta(\log r)}.$$

Since $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$, the proof of Proposition 7 would run parallelly to the one of Proposition 6. We omit the details.

2. Main Results

In this paper, our aim is to make use of the concept of (α,β)-order of entire functions to investigate distribution of zeros and growth of solutions of equation (1), which considerably extends some results of [21].

Theorem 8. *Let $A(z)$ be an entire function satisfying $\sigma_{(\alpha,\beta)}[A] > 0$. Then $\sigma_{(\alpha(\log),\beta)}[f] = \sigma_{(\alpha,\beta)}[A]$ holds for all non-trivial solutions of (1).*

Remark 9. If we choose $\alpha(r) = \log^{[p-1]} r$ $(p \geq 2)$ and $\beta(r) = r$ in Theorem 8, we obtain Theorem 3.1 in [17] for $p \geq 2$. Furthermore, by setting $\alpha(r) = \log^{[p-1]} r$ $(p \geq 2)$ and $\beta(r) = \log^{[q]} \varphi(e^r)$ $(q \geq 1)$ in Theorem 8, we obtain Theorem 2.1 in [21] for $p \geq q \geq 2$ and $p = 2$, $q = 1$. We assume that $\varphi : [0,+\infty) \to (0,+\infty)$ is a non-decreasing unbounded function and always satisfies the following two conditions:

(i) $\lim\limits_{r\to+\infty} \frac{\log_{p+1} r}{\log_q \varphi(r)} = 0$.

(ii) $\lim\limits_{r\to+\infty} \frac{\log_q \varphi(\eta r)}{\log_q \varphi(r)} = 1$ for some $\eta > 1$.

Theorem 10. *Let $A(z)$ be an entire function satisfying $\sigma_{(\alpha,\beta)}[A] > 0$, let f_1 and f_2 be two linearly independent solutions of (1) and denote $F = f_1 \cdot f_2$. Then*

$$\max\{\lambda_{(\alpha(\log),\beta)}[f_1], \lambda_{(\alpha(\log),\beta)}[f_2]\} = \lambda_{(\alpha(\log),\beta)}[F] = \sigma_{(\alpha(\log),\beta)}[F] \leq \sigma_{(\alpha,\beta)}[A].$$

If $\sigma_{(\alpha(\log),\beta)}[F] < \sigma_{(\alpha,\beta)}[A]$, then $\lambda_{(\alpha(\log),\beta)}[f] = \sigma_{(\alpha,\beta)}[A]$ holds for all solutions of type $f = c_1 f_1 + c_2 f_2$, where $c_1 \cdot c_2 \neq 0$.

Remark 11. By setting $\alpha(r) = \log^{[p-1]} r$ $(p \geq 2)$ and $\beta(r) = r$ in Theorem 10, we obtain Theorem 3.2 in [17] for $p \geq 2$. Moreover, by putting $\alpha(r) = \log^{[p-1]} r$ $(p \geq 2)$ and $\beta(r) = \log^{[q]} \varphi(e^r)$ $(q \geq 1)$ in Theorem 10 for $p \geq q \geq 2$ and $p = 2$, $q = 1$, where $\varphi(r)$ satisfies the two conditions in Remark 9, we obtain Theorem 2.2 in [21].

Theorem 12. *Let $A(z)$ be an entire function satisfying $\overline{\lambda}_{(\alpha,\beta)}[A] < \sigma_{(\alpha,\beta)}[A]$. Then $\lambda_{(\alpha(\log),\beta)}[f] \leq \sigma_{(\alpha,\beta)}[A] \leq \lambda_{(\alpha,\beta)}[f]$ holds for all non-trivial solutions of* (1).

Remark 13. If we put $\alpha(r) = \log^{[p-1]} r$ $(p \geq 2)$ and $\beta(r) = r$ in Theorem 12, we obtain Theorem 3.3 in [17] for $p \geq 2$. Furthermore, by choosing $\alpha(r) = \log^{[p-1]} r$ $(p \geq 2)$ and $\beta(r) = \log^{[q]} \varphi(e^r)$ $(q \geq 1)$ in Theorem 12 for $p \geq q \geq 2$ and $p = 2$, $q = 1$, where $\varphi(r)$ satisfies the two conditions in Remark 9, we obtain Theorem 2.3 in [21].

3. Some Lemmas

In this section, we present the following lemmas which will be needed in the sequel.

Lemma 14. *([12, 13, 18]) Let f be a transcendental entire function, and let z be a point with $|z| = r$ at which $|f(z)| = M(r, f)$. Then, for all $|z|$ outside a set E_1 of r of finite logarithmic measure, we have*

$$\frac{f^{(j)}(z)}{f(z)} = \Big(\frac{\nu(r,f)}{z}\Big)^j (1 + o(1)) \quad (j \in \mathbb{N}), \tag{10}$$

where $\nu(r, f)$ is the central index of f.

Lemma 15. *([9, 10, 18]) Let $g : [0, +\infty) \to \mathbb{R}$ and $h : [0, +\infty) \to \mathbb{R}$ be monotone nondecreasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E_2 of finite linear measure or finite logarithmic measure. Then, for any $d > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(dr)$ for all $r > r_0$.*

Lemma 16. *([13], Theorems 1.9 and 1.10, or [16], Satz 4.3 and 4.4) Let* $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ *be any entire function,* $\mu(r,f)$ *be the maximum term, i.e.,* $\mu(r,f) = \max\{|a_n|r^n; n = 0,1,...\}$*, and* $\nu(r,f)$ *be the central index of* f*.*

(i) If $|a_0| \neq 0$*, then*

$$\log \mu(r,f) = \log|a_0| + \int_0^r \frac{\nu(t,f)}{t} dt. \tag{11}$$

(ii) For $r < R$*, we have*

$$M(r,f) < \mu(r,f)\Big(\nu(R,f) + \frac{R}{R-r}\Big). \tag{12}$$

Lemma 17. *Let* f *be an entire function satisfying* $\sigma_{(\alpha,\beta)}[f] = \sigma_1$ *and* $\mu_{(\alpha,\beta)}[f] = \mu_1$*, and let* $\nu(r,f)$ *be the central index of* f*. Then*

$$\limsup_{r\to+\infty} \frac{\alpha(\log \nu(r,f))}{\beta(\log r)} = \sigma_1 \ \ and \ \ \liminf_{r\to+\infty} \frac{\alpha(\log \nu(r,f))}{\beta(\log r)} = \mu_1.$$

Proof. In view of the first part of Lemma 16, one may obtain that (cf. [5])

$$\log \mu(2r,f) = \log|a_0| + \int_0^{2r} \frac{\nu(t,f)}{t} dt$$

$$\geq \log|a_0| + \int_r^{2r} \frac{\nu(t,f)}{t} dt \geq \log|a_0| + \nu(r,f)\log 2. \tag{13}$$

Also, by Cauchy's inequality, it is well known that (cf. [24])

$$\mu(r,f) \leq M(r,f). \tag{14}$$

Therefore one may obtain from (13) and (14) that (cf. [5])

$$\nu(r,f)\log 2 \leq \log M(2r,f) - \log|a_0|.$$

Thus, from above we get that

$$\log \nu(r,f) + \log^{[2]} 2 \leq \log^{[2]} M(2r,f) + \log\Big(1 - \frac{\log|a_0|}{\log M(2r,f)}\Big),$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha((1+o(1))\log\nu(r,f))}{\beta(\log r)}\leq\limsup_{r\to+\infty}\frac{\alpha((1+o(1))\log^{[2]}M(2r,f))}{\beta(\log 2r-\log 2)},$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{(1+o(1))\alpha(\log\nu(r,f))}{\beta(\log r)}\leq\limsup_{r\to+\infty}\frac{(1+o(1))\alpha(\log^{[2]}M(2r,f))}{\beta((1+o(1))\log 2r)},$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha(\log\nu(r,f))}{\beta(\log r)}\leq\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}M(2r,f))}{(1+o(1))\beta(\log 2r)},$$

$$i.e.,\ \sigma_1=\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}M(2r,f))}{\beta(\log 2r)}\geq\limsup_{r\to+\infty}\frac{\alpha(\log\nu(r,f))}{\beta(\log r)},\tag{15}$$

and consequently

$$\mu_1\geq\liminf_{r\to+\infty}\frac{\alpha(\log\nu(r,f))}{\beta(\log r)}.\tag{16}$$

Further, for any constant K_1 one may get from the second part of Lemma 16, that (cf. [5])

$$\log M(r,f)<\nu(r,f)\log r+\log\nu(2r,f)+K_1.$$

Therefore from above we obtain that

$$\begin{aligned}\log M(r,f)&<\nu(2r,f)\log r+\nu(2r,f)+K_1,\\ i.e.,\ \log M(r,f)&<\nu(2r,f)(1+\log r)+K_1,\\ i.e.,\ \log M(r,f)&<\nu(2r,f)\log(e\cdot r)+K_1,\end{aligned}$$

$$i.e.,\ \log^{[2]}M(r,f)<\log\nu(2r,f)+\log^{[2]}(e\cdot r)+\log\left(1+\frac{K_1}{\nu(2r,f)\log(e\cdot r)}\right),$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}M(r,f))}{\beta(\log r)}\leq\limsup_{r\to+\infty}\frac{\alpha((1+o(1))\log\nu(2r,f))}{\beta(\log r)},$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}M(r,f))}{\beta(\log r)}\leq\limsup_{r\to+\infty}\frac{(1+o(1))\alpha(\log\nu(r,f))}{\beta(\log r-\log 2)},$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}M(r,f))}{\beta(\log r)}\leq\limsup_{r\to+\infty}\frac{\alpha(\log\nu(r,f))}{\beta((1+o(1))\log r)},$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}M(r,f))}{\beta(\log r)}\leq\limsup_{r\to+\infty}\frac{\alpha(\log\nu(r,f))}{(1+o(1))\beta(\log r)},$$

$$i.e.,\ \sigma_1=\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}M(r,f))}{\beta(\log r)}\leq\limsup_{r\to+\infty}\frac{\alpha(\log\nu(r,f))}{\beta(\log r)},\tag{17}$$

and accordingly

$$\mu_1 \leq \liminf_{r\to+\infty} \frac{\alpha(\log \nu(r,f))}{\beta(\log r)}. \tag{18}$$

Combining (15), (17) and (16), (18) we obtain that

$$\limsup_{r\to+\infty} \frac{\alpha(\log \nu(r,f))}{\beta(\log r)} = \sigma_1 \text{ and } \liminf_{r\to+\infty} \frac{\alpha(\log \nu(r,f))}{\beta(\log r)} = \mu_1.$$

This proves the lemma. □

Lemma 18. *Let f be an entire function satisfying*

$$\sigma_{(\alpha(\log),\beta)}[f] = \sigma_2 \quad \textit{and} \quad \mu_{(\alpha(\log),\beta)}[f] = \mu_2,$$

and let $\nu(r,f)$ be the central index of f. Then

$$\limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} \nu(r,f))}{\beta(\log r)} = \sigma_2 \quad \textit{and} \quad \liminf_{r\to+\infty} \frac{\alpha(\log^{[2]} \nu(r,f))}{\beta(\log r)} = \mu_2.$$

The proof of Lemma 18 can be conducted along the same lines as the proof of Lemma 17 and so it is omitted.

Lemma 19. *Let f_1 and f_2 be the entire functions of (α,β)-exponent of convergence of the zero sequence and denote $F = f_1 \cdot f_2$. Then*

$$\lambda_{(\alpha,\beta)}[F] = \max\{\lambda_{(\alpha,\beta)}[f_1],\ \lambda_{(\alpha,\beta)}[f_2]\}.$$

Proof. Let $n(r,0,F), n(r,0,f_1)$ and $n(r,0,f_2)$ be the unintegrated counting functions for the number of zeros of F, f_1 and f_2. For any $r > 0$, it is easy to see that

$$n(r,0,F) \geq \max\{n(r,0,f_1),\ n(r,0,f_2)\}. \tag{19}$$

By Definition 3 and (19), we have

$$\lambda_{(\alpha,\beta)}[F] \geq \max\{\lambda_{(\alpha,\beta)}[f_1],\ \lambda_{(\alpha,\beta)}[f_2]\}. \tag{20}$$

On the other hand, since the zeros of F must be the zeros of f_1 and the zeros of f_2, for any $r > 0$ we have

$$n(r,0,F) = n(r,0,f_1) + n(r,0,f_2) \leq 2\max\{n(r,0,f_1),\ n(r,0,f_2)\}. \tag{21}$$

By Definition 3 and (21), we get that

$$\lambda_{(\alpha,\beta)}[F] \leq \max\{\lambda_{(\alpha,\beta)}[f_1],\ \lambda_{(\alpha,\beta)}[f_2]\}. \tag{22}$$

Therefore, by (20) and (22), we have

$$\lambda_{(\alpha,\beta)}[F] = \max\{\lambda_{(\alpha,\beta)}[f_1],\ \lambda_{(\alpha,\beta)}[f_2]\}.$$

This completes the proof. □

Lemma 20. *Let f_1 and f_2 be the entire functions of $(\alpha(\log),\beta)$-exponent of convergence of the zero sequence and denote $F = f_1 \cdot f_2$. Then*

$$\lambda_{(\alpha(\log),\beta)}[F] = \max\{\lambda_{(\alpha(\log),\beta)}[f_1],\ \lambda_{(\alpha(\log),\beta)}[f_2]\}.$$

The proof of Lemma 20 can be conducted along the same lines as the proof of Lemma 19 and so this proof is omitted.

Lemma 21. *Let f be a transcendental meromorphic function satisfying $\sigma_{(\alpha,\beta)}[f] = \sigma_3$ and let $k \geq 1$ be an integer. Then, for any $\varepsilon > 0$, there exists a set E_3, having finite linear measure, such that for all $r \notin E_3$ we have*

$$m\Big(r, \frac{f^{(k)}}{f}\Big) = O\left(\alpha^{-1}\left(\left(\sigma_3 + \varepsilon\right)\beta\left(\log r\right)\right)\right).$$

Proof. Set $k = 1$. Since $\sigma_{(\alpha,\beta)}[f] = \sigma_3 < +\infty$, for sufficiently large r and for any given $\varepsilon > 0$, we have

$$T(r,f) < \exp\left(\alpha^{-1}\left(\left(\sigma_3 + \varepsilon\right)\beta\left(\log r\right)\right)\right). \tag{23}$$

By the lemma of logarithmic derivative, we have

$$m\Big(r, \frac{f'}{f}\Big) = O(\log r + \log T(r,f)) \quad (r \notin E_3), \tag{24}$$

where $E_3 \subset [0,+\infty)$ is a set of finite linear measure, not necessarily the same at each occurrence. By (23) and (24) and the condition $\alpha(\log x) = o(\beta(x))$ as $x \to +\infty$, we have

$$m\Big(r, \frac{f'}{f}\Big) = O\left(\alpha^{-1}\left(\left(\sigma_3 + \varepsilon\right)\beta\left(\log r\right)\right)\right) \quad (r \notin E_3).$$

We assume that

$$m\Big(r, \frac{f^{(k)}}{f}\Big) = O\left(\alpha^{-1}\left(\left(\sigma_3+\varepsilon\right)\beta\left(\log r\right)\right)\right) \quad (r \notin E_3) \tag{25}$$

holds for a certain integer $k \geq 1$. By $N(r, f^{(k)}) \leq (k+1)N(r, f)$, for all $r \notin E_3$, we have

$$\begin{aligned} T(r, f^{(k)}) &= m(r, f^{(k)}) + N(r, f^{(k)}) \\ &\leq m\Big(r, \frac{f^{(k)}}{f}\Big) + m(r, f) + (k+1)N(r, f) \\ &\leq (k+1)T(r, f) + O\left(\alpha^{-1}\left(\left(\sigma_3+\varepsilon\right)\beta\left(\log r\right)\right)\right). \end{aligned} \tag{26}$$

By (24) and (26), for $r \notin E_3$, we obtain that

$$\begin{aligned} m\Big(r, \frac{f^{(k+1)}}{f^{(k)}}\Big) = m\Big(r, \frac{\left(f^{(k)}\right)'}{f^{(k)}}\Big) = O(\log r + \log T(r, f^{(k)})) \\ = O\left(\alpha^{-1}\left(\left(\sigma_3+\varepsilon\right)\beta\left(\log r\right)\right)\right). \end{aligned} \tag{27}$$

Therefore, by (25) and (27), for $r \notin E_3$, we get that

$$\begin{aligned} m\Big(r, \frac{f^{(k+1)}}{f}\Big) &\leq m\Big(r, \frac{f^{(k+1)}}{f^{(k)}}\Big) + m\Big(r, \frac{f^{(k)}}{f}\Big) \\ &= O\left(\alpha^{-1}\left(\left(\sigma_3+\varepsilon\right)\beta\left(\log r\right)\right)\right). \end{aligned}$$

Hence the lemma follows. □

4. Proof of the Main Results

Proof of Theorem 8. Set $\sigma_{(\alpha,\beta)}[A] = \sigma_4 > 0$. First, we prove that every solution of (1) satisfies $\sigma_{(\alpha(\log),\beta)}[f] \leq \sigma_4$. If f is a polynomial solution of (1), it is easy to show that $\sigma_{(\alpha(\log),\beta)}[f] = 0 \leq \sigma_4$ holds. Suppose that f is a transcendental solution of (1). By (1), we can write

$$\left|\frac{f''(z)}{f(z)}\right| = \left|A(z)\right|,$$

so, by Lemma 14, there exists a set $E_1 \subset (1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$ and $|f(z)| = M(r, f)$, we have

$$\Big(\frac{\nu(r,f)}{r}\Big)^2 |1 + o(1)| \le \exp^{[2]} \Big(\alpha^{-1}\Big(\Big(\sigma_4 + \frac{\varepsilon}{2}\Big)\beta(\log r)\Big)\Big),$$

and hence, we obtain that

$$\nu(r, f) \le r \exp^{[2]}(\alpha^{-1}((\sigma_4 + \varepsilon)\beta(\log r))) \quad (r \notin E_1). \tag{28}$$

Therefore by (28) and Lemma 15, there exists some $\eta_1 > 1$ such that for all $r > r_0$ we have

$$\nu(r, f) \le \eta_1 r \exp^{[2]}(\alpha^{-1}((\sigma_4 + \varepsilon)\beta(\log \eta_1 r))). \tag{29}$$

By (29), Lemma 18, and the two conditions on α and β, we obtain that

$$\sigma_{(\alpha(\log),\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} \nu(r, f))}{\beta(\log r)} \le \sigma_4. \tag{30}$$

On the other hand, by (1), since f is a transcendental, we get that

$$m(r, A) = m\Big(r, -\frac{f''}{f}\Big) = O(\log rT(r, f))$$

$$= O(\log r + \log T(r, f)),\ (r \notin E_3),$$

where $E_3 \subset [0, +\infty)$ is a set of finite linear measure. By using Lemma 15, for any $\eta_2 > 1$ and for all $r > r_0$, we have

$$m(r, A) = m\Big(r, -\frac{f''}{f}\Big) \le K_2(\log \eta_2 r + \log T(\eta_2 r, f)), \tag{31}$$

where $K_2 > 0$ is some constant. By (31), by using the two inequalities $\log(x + y) \le \log x + \log y + \log 2$ $(x,\ y \ge 1)$ and $x + y \le 2 \max\{x, y\}$, since $A(z)$ is an entire function, we have

$$\sigma_{(\alpha,\beta)}[A] = \limsup_{r\to+\infty} \frac{\alpha(\log m(r, A))}{\beta(\log r)}$$

$$\le \limsup_{r\to+\infty} \frac{\alpha(\log K_2 + \log\log \eta_2 r + \log\log T(\eta_2 r, f) + \log 2)}{\beta(\log r)}$$

$$\le \limsup_{r\to+\infty} \frac{\alpha((1 + o(1)) (\log\log \eta_2 r + \log\log T(\eta_2 r, f)))}{\beta(\log r)}$$

$$= \limsup_{r\to+\infty} \frac{(1+o(1))\,\alpha(\log\log\eta_2 r + \log\log T(\eta_2 r, f))}{\beta(\log r)}$$

$$\leq \limsup_{r\to+\infty} \frac{\alpha(2\max\{\log\log\eta_2 r, \log\log T(\eta_2 r, f)\})}{\beta(\log r)}$$

$$\leq \limsup_{r\to+\infty} \frac{(1+o(1))\max\{\alpha(\log\log\eta_2 r), \alpha(\log\log T(\eta_2 r, f))\}}{\beta(\log r)}$$

$$\leq \limsup_{r\to+\infty} \frac{\alpha(\log\log\eta_2 r) + \alpha(\log\log T(\eta_2 r, f))}{\beta(\log r)}$$

$$\leq \limsup_{r\to+\infty} \frac{\alpha(\log\log\eta_2 r)}{\beta(\log\eta_2 r - \log\eta_2)} + \limsup_{r\to+\infty} \frac{\alpha(\log\log T(\eta_2 r, f))}{\beta(\log\eta_2 r - \log\eta_2)}$$

$$\leq \limsup_{r\to+\infty} \frac{\alpha(\log\log\eta_2 r)}{(1+o(1))\,\beta(\log\eta_2 r)} + \limsup_{r\to+\infty} \frac{\alpha(\log\log T(\eta_2 r, f))}{(1+o(1))\,\beta(\log\eta_2 r)} = \sigma_{(\alpha(\log),\beta)}[f],$$

since $\alpha(\log x) = o(\beta(x))$ as $x \to +\infty$ we have $\frac{\alpha(\log^{[2]}\eta_2 r)}{\beta(\log\eta_2 r)} = \frac{\alpha(\log^{[2]} R)}{\beta(\log R)} \to 0$ as $R = \eta_2 r \to +\infty$. Therefore, we get that $\sigma_{(\alpha(\log),\beta)}[f] = \sigma_{(\alpha,\beta)}[A]$ holds for all non-trivial solutions of (1). Thus Theorem 8 follows.

Proof of Theorem 10. Set $\sigma_{(\alpha,\beta)}[A] = \sigma_5 > 0$, by Theorem 8 we have $\sigma_{(\alpha(\log),\beta)}[f_1] = \sigma_{(\alpha(\log),\beta)}[f_2] = \sigma_{(\alpha,\beta)}[A] = \sigma_5$. Hence, we have

$$\lambda_{(\alpha(\log),\beta)}[F] \leq \sigma_{(\alpha(\log),\beta)}[F]$$

$$\leq \max\{\sigma_{(\alpha(\log),\beta)}[f_1], \sigma_{(\alpha(\log),\beta)}[f_2]\} = \sigma_{(\alpha,\beta)}[A]. \tag{32}$$

By (32) and Lemma 20, we have

$$\max\{\lambda_{(\alpha(\log),\beta)}[f_1], \lambda_{(\alpha(\log),\beta)}[f_2]\} = \lambda_{(\alpha(\log),\beta)}[F]$$

$$\leq \sigma_{(\alpha(\log),\beta)}[F] \leq \sigma_{(\alpha,\beta)}[A]. \tag{33}$$

It remains to show that $\lambda_{(\alpha(\log),\beta)}[F] = \sigma_{(\alpha(\log),\beta)}[F]$. By (1), we have (see [17], [18, pp. 76-77]) that all zeros of F are simple and that

$$F^2 = C^2\left(\left(\frac{F'}{F}\right)^2 - 2\left(\frac{F''}{F}\right) - 4A\right)^{-1}, \tag{34}$$

where $C \neq 0$ is a constant. Hence,

$$\begin{aligned} 2T(r,F) &= T\Big(r, \Big(\frac{F'}{F}\Big)^2 - 2\Big(\frac{F''}{F}\Big) - 4A\Big) + O(1) \\ &\leq O\Big(\overline{N}\Big(r, \frac{1}{F}\Big) + m\Big(r, \frac{F'}{F}\Big) + m\Big(r, \frac{F''}{F}\Big) + m(r,A)\Big). \end{aligned} \tag{35}$$

By $\sigma_{(\alpha(\log),\beta)}[f] = \sigma_{(\alpha,\beta)}[A] = \sigma_5 < +\infty$ and Lemma 21, for all $r \notin E_3$, we have

$$\begin{aligned} m(r,A) = m\Big(r, \frac{f''}{f}\Big) &= O(\exp(\alpha^{-1}((\sigma_5+\varepsilon)\beta(\log r)))), \\ m\Big(r, \frac{F'}{F}\Big) &= O(\exp(\alpha^{-1}((\sigma_5+\varepsilon)\beta(\log r)))), \\ m\Big(r, \frac{F''}{F}\Big) &= O(\exp(\alpha^{-1}((\sigma_5+\varepsilon)\beta(\log r)))). \end{aligned}$$

Therefore, by (35), for all $r \notin E_3$ we have

$$T(r,F) = O\Big(\overline{N}\Big(r, \frac{1}{F}\Big) + \exp(\alpha^{-1}((\sigma_5+\varepsilon)\beta(\log r)))\Big). \tag{36}$$

Now, let us assume that $\lambda_{(\alpha(\log),\beta)}[F] < \kappa < \sigma_{(\alpha(\log),\beta)}[F]$. Since all zeros of F are simple, we obtain

$$\overline{N}\Big(r, \frac{1}{F}\Big) = N\Big(r, \frac{1}{F}\Big) = O(\exp^{[2]}(\alpha^{-1}(\kappa\beta(\log r)))). \tag{37}$$

Hence by (36) and (37), for all $r \notin E_3$, we get that

$$T(r,F) = O(\exp^{[2]}(\alpha^{-1}(\kappa\beta(\log r)))).$$

By Definition 1 and Lemma 15, we have $\sigma_{(\alpha(\log),\beta)}[F] \leq \kappa < \sigma_{(\alpha(\log),\beta)}[F]$, this is a contradiction. Therefore, the first assertion is proved.

If $\sigma_{(\alpha(\log),\beta)}[F] < \sigma_{(\alpha,\beta)}[A]$, let us assume that $\lambda_{(\alpha(\log),\beta)}[f] < \sigma_{(\alpha,\beta)}[A]$ holds for any solution of type $f = c_1 f_1 + c_2 f_2$ $(c_1 c_2 \neq 0)$. We denote $F = f_1 \cdot f_2$ and $F_1 = f \cdot f_1$, then we have $\lambda_{(\alpha(\log),\beta)}[F] < \sigma_{(\alpha,\beta)}[A]$ and $\lambda_{(\alpha(\log),\beta)}[F_1] < \sigma_{(\alpha,\beta)}[A]$. Since (36) holds for F and F_1, $F_1 = f \cdot f_1 = (c_1 f_1 + c_2 f_2) f_1 = c_1 f_1^2 + c_2 F$, then we obtain

$$\begin{aligned} T(r,f_1) &= O(T(r,F_1) + T(r,F)) = O\Big(\overline{N}\Big(r, \frac{1}{F_1}\Big) \\ &\quad + \overline{N}\Big(r, \frac{1}{F}\Big) + \exp(\alpha^{-1}((\sigma_5+\varepsilon)\beta(\log r)))\Big). \end{aligned} \tag{38}$$

By $\lambda_{(\alpha(\log),\beta)}[F] < \sigma_{(\alpha,\beta)}[A]$, $\lambda_{(\alpha(\log),\beta)}[F_1] < \sigma_{(\alpha,\beta)}[A]$ and (37), for some $\kappa < \sigma_{(\alpha,\beta)}[A]$, we get that

$$T(r, f_1) = O(\exp^{[2]}(\alpha^{-1}(\kappa\beta(\log r)))). \tag{39}$$

By Definition 1 and (39), we have $\sigma_{(\alpha(\log),\beta)}[f_1] \leq \kappa < \sigma_{(\alpha,\beta)}[A]$, this is a contradiction with Theorem 8. Therefore, we have that $\lambda_{(\alpha(\log),\beta)}[f] = \sigma_{(\alpha,\beta)}[A]$ holds for all solutions of type $f = c_1 f_1 + c_2 f_2$, where $c_1 c_2 \neq 0$. Hence the theorem follows.

Proof of Theorem 12. By Theorem 8 and $\lambda_{(\alpha(\log),\beta)}[f] \leq \sigma_{(\alpha(\log),\beta)}[f]$, it is easy to show that $\lambda_{(\alpha(\log),\beta)}[f] \leq \sigma_{(\alpha,\beta)}[A]$ holds. It remains to show that $\sigma_{(\alpha,\beta)}[A] \leq \lambda_{(\alpha,\beta)}[f]$. Let us assume that $\sigma_{(\alpha,\beta)}[A] > \lambda_{(\alpha,\beta)}[f]$. By (1) and a similar proof of Theorem 5.6 in [18, pp. 82], we obtain

$$T\Big(r, \frac{f}{f'}\Big) = O\Big(\overline{N}\Big(r, \frac{1}{f}\Big) + \overline{N}\Big(r, \frac{1}{A}\Big)\Big) \quad (r \notin E_3). \tag{40}$$

By (40) and the assumption $\sigma_{(\alpha,\beta)}[A] > \lambda_{(\alpha,\beta)}[f]$ and $\overline{\lambda}_{(\alpha,\beta)}[A] < \sigma_{(\alpha,\beta)}[A]$, we get for some $\kappa < \sigma_{(\alpha,\beta)}[A]$ that

$$T\Big(r, \frac{f}{f'}\Big) = O(\exp(\alpha^{-1}(\kappa\beta(\log r)))). \tag{41}$$

Further, by Definition 1 and (41), we have $\sigma_{(\alpha,\beta)}\Big[\frac{f}{f'}\Big] = \sigma_{(\alpha,\beta)}\Big[\frac{f'}{f}\Big] \leq \kappa < \sigma_{(\alpha,\beta)}[A]$. Therefore by

$$-A(z) = \Big(\frac{f'}{f}\Big)' + \Big(\frac{f'}{f}\Big)^2,$$

we get that $\sigma_{(\alpha,\beta)}[A] \leq \sigma_{(\alpha,\beta)}\Big[\frac{f'}{f}\Big] < \sigma_{(\alpha,\beta)}[A]$, which is a contradiction. Hence, we have that $\lambda_{(\alpha(\log),\beta)}[f] \leq \sigma_{(\alpha,\beta)}[A] \leq \lambda_{(\alpha,\beta)}[f]$ holds for all non-trivial solutions of (1). The proof is complete.

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