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HOMOTOPY PERTURBATION METHOD IN THE HEAT CONDUCTION PROBLEMS

Summary. In this paper an application of the homotopy perturbation method for solving the steady state and unsteady state heat conduction problem is presented.

HOMOTOPIJNA METODA PERTURBACYJNA W ZAGADNIENIACH PRZEWODZENIA CIEPŁA

Streszczenie. W artykule przedstawiono zastosowanie homotopijnej metody perturbacyjnej do rozwiązania zagadnień ustalonego oraz nieustalonego przewodzenia ciepła.

1. Introduction

Homotopy perturbation method arised as the connection of elements of two other methods, namely, the homotopy analysis method [1, 7, 10] and the perturbation method [3, 8, 12]. Its inventor was the Chinese mathematician Ji-Huan

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He [5, 6, 8, 9, 11, 13, 14]. Homotopy perturbation method enables to seek the solution of the operator equation

$$A(u) = f(z), \qquad z \in \Omega, \tag{1}$$

where A denotes the operator, f is the known function, and u represents the sought function. Operator A is presented in form of the following sum

$$A(u) = L(u) + N(u), \tag{2}$$

where L is the linear operator, whereas N denotes the non-linear operator. Thus, equation (1) can be written in the form

$$L(u) + N(u) = f(z), \qquad z \in \Omega. \tag{3}$$

Let us define a new operator H, called as the homotopy operator, in the following way

$$H(v,p) := (1-p)\big(L(v) - L(u_0)\big) + p\,\big(A(v) - f(z)\big), \tag{4}$$

where $p \in [0,1]$ denotes the, so called, homotopy parameter, $v(z,p) : \Omega \times [0,1] \to \mathbb{R}$, and u_0 describes the initial approximation of the solution of equation (1). By using the relation (2) we receive

$$H(v,p) = L(v) - L(u_0) + p\,L(u_0) + p\,\big(N(v) - f(z)\big). \tag{5}$$

Since $H(v,0) = L(v) - L(u_0)$, therefore, for $p = 0$, solving the operator equation $H(v,0) = 0$ is equivalent to solving the trivial problem $L(v) - L(u_0) = 0$. Whereas, for $p = 1$, solving the operator equation $H(v,1) = 0$ is equivalent to solving the equation (1). Thus, the monotonic change of parameter p, between zero and one, corresponds with the continuous change between the trivial equation $L(v) - L(u_0) = 0$ and the considered equation (it means, with the continuous change of the solution v between u_0 and u).

Solution of the equation $H(v,p) = 0$ is sought in form of the power series

$$v = \sum_{j=0}^{\infty} p^j\, v_j. \tag{6}$$

If the above series is convergent then, by substituting $p = 1$, we obtain the solution of equation (1):

$$u = \lim_{p \to 1} v = \sum_{j=0}^{\infty} v_j. \tag{7}$$

Information about convergence of the series (6) can be found in papers [2, 6]. In many cases the series (6) is rapidly convergent, therefore, reducing the above sum to the few initial components may assure to receive a very good approximation of the solution. If we reduce the sum to the first $n+1$ components, we receive the, so called, n-order approximate solution

$$\widehat{u}_n = \sum_{j=0}^{n} v_j. \tag{8}$$

In order to find the function v_j we substitute relation (6) into the equation $H(v,p) = 0$ and we compare the expressions with the same powers of parameter p. In this way, we receive the sequence of operator equations enabling to determine the successive functions v_j. By these means, finding the solution of considered problem can be reduced to solving the sequence of problems, solutions of which are easy to determine.

2. Steady state heat conduction

Let us introduce an application of the considered method for solving the steady state heat conduction problem described with the aid of Laplace equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0, \qquad (x,y) \in D, \tag{9}$$

where region D is the rectangle $(b_1, b_2) \times (d_1, d_2)$. On the boundary of the region the boundary conditions of the first kind are given

$$u(b_1, y) = \varphi_1(y), \qquad u(b_2, y) = \varphi_2(y), \tag{10}$$

$$u(x, d_1) = \theta_1(x), \qquad u(x, d_2) = \theta_2(x). \tag{11}$$

In case of the considered Laplace equation, we can apply the averaging method, similarly as it is done for the Adomian decomposition method [4]. In this method, in our case, we solve two problems with the various selection of the linear operator ($L = \frac{\partial^2}{\partial x^2}$ or $L = \frac{\partial^2}{\partial y^2}$), averaged solutions of which give the solution of considered problem.

Thus, we begin by defining two equivalent homotopy operators for equation (9) having the following form

$$H_1(v_1,p) := \frac{\partial^2 v_1}{\partial x^2} - \frac{\partial^2 u_{1,0}}{\partial x^2} + p\left(\frac{\partial^2 u_{1,0}}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2}\right), \tag{12}$$

$$H_2(v_2,p) := \frac{\partial^2 v_2}{\partial y^2} - \frac{\partial^2 u_{2,0}}{\partial y^2} + p\left(\frac{\partial^2 u_{2,0}}{\partial y^2} + \frac{\partial^2 v_2}{\partial x^2}\right). \tag{13}$$

Solutions of equations ($i = 1, 2$):

$$H_i(v_i, p) = 0 \tag{14}$$

will be sought in the form of power series of the variable p:

$$v_i = \sum_{j=0}^{\infty} p^j \, v_{i,j}. \tag{15}$$

By substituting the relations (15) into the equations (12) and (13), after some transformations, we get ($i = 1, 2$):

$$\sum_{j=0}^{\infty} p^j \, \frac{\partial^2 v_{1,j}}{\partial x^2} = \frac{\partial^2 u_{1,0}}{\partial x^2} - p\,\frac{\partial^2 u_{1,0}}{\partial x^2} - \sum_{j=1}^{\infty} p^j \, \frac{\partial^2 v_{1,j-1}}{\partial y^2} \tag{16}$$

and

$$\sum_{j=0}^{\infty} p^j \, \frac{\partial^2 v_{2,j}}{\partial y^2} = \frac{\partial^2 u_{2,0}}{\partial y^2} - p\,\frac{\partial^2 u_{2,0}}{\partial y^2} - \sum_{j=1}^{\infty} p^j \, \frac{\partial^2 v_{2,j-1}}{\partial x^2}. \tag{17}$$

Now, by comparing the expressions with the same powers of parameter p we obtain the following systems of equations

$$\begin{cases} \dfrac{\partial^2 v_{1,0}}{\partial x^2} = \dfrac{\partial^2 u_{1,0}}{\partial x^2}, \\[2mm] \dfrac{\partial^2 v_{2,0}}{\partial y^2} = \dfrac{\partial^2 u_{2,0}}{\partial y^2}, \end{cases} \tag{18}$$

$$\begin{cases} \dfrac{\partial^2 v_{1,1}}{\partial x^2} = -\dfrac{\partial^2 u_{1,0}}{\partial x^2} - \dfrac{\partial^2 v_{1,0}}{\partial y^2}, \\[2mm] \dfrac{\partial^2 v_{2,1}}{\partial y^2} = -\dfrac{\partial^2 u_{2,0}}{\partial y^2} - \dfrac{\partial^2 v_{2,0}}{\partial x^2}, \end{cases} \tag{19}$$

and for $j \geqslant 2$:

$$\begin{cases} \dfrac{\partial^2 v_{1,j}}{\partial x^2} = -\dfrac{\partial^2 v_{1,j-1}}{\partial y^2}, \\[2mm] \dfrac{\partial^2 v_{2,j}}{\partial y^2} = -\dfrac{\partial^2 v_{2,j-1}}{\partial x^2}. \end{cases} \tag{20}$$

The above systems of partial differential equations must be completed by the conditions ensuring the uniqueness of solution of those systems. For the first system (18) we define the conditions

$$\begin{cases} v_{1,0}(b_1, y) = \varphi_1(y), \\ v_{1,0}(b_2, y) = \varphi_2(y), \\ v_{2,0}(x, d_1) = \theta_1(x), \\ v_{2,0}(x, d_2) = \theta_2(x), \end{cases} \tag{21}$$

whereas, for the other systems ($j \geqslant 1$) we determine conditions of the form

$$\begin{cases} v_{1,j}(b_1, y) = 0, \\ v_{1,j}(b_2, y) = 0, \\ v_{2,j}(x, d_1) = 0, \\ v_{2,j}(x, d_2) = 0. \end{cases} \tag{22}$$

Afterwards, the sought solution is given by the averaged function

$$u(x, y) = \frac{v_1(x, y) + v_2(x, y)}{2} = \frac{1}{2} \sum_{j=0}^{\infty} \big(v_{1,j}(x, y) + v_{2,j}(x, y) \big). \tag{23}$$

Example 2.1

Application of the proposed method will be illustrated by the example in which: $b_1 = 1$, $b_2 = \pi$, $d_1 = 0$, $d_2 = \pi$ and

$$\begin{aligned} \varphi_1(y) &= \sinh(1) \cos(y), \\ \varphi_2(y) &= \sinh(\pi) \cos(y), \\ \theta_1(x) &= \sinh(x), \\ \theta_2(x) &= -\sinh(x). \end{aligned}$$

Exact solution of the above problem is given by the function

$$u(x, y) = \sinh(x) \cos(y).$$

As the initial approximations $u_{1,0}$ and $u_{2,0}$ we take the zero functions

$$u_{1,0}(x, y) = u_{2,0}(x, y) = 0.$$

By solving the proper systems of equations we obtain, successively

$$v_{1,0}(x,t) = \frac{1}{\pi - 1} \cos(y) \big((\pi - x) \sinh(1) + (x-1) \sinh(\pi)\big),$$
$$v_{2,0}(x,t) = \Big(1 - \frac{2\,y}{\pi}\Big) \sinh(x)$$

and

$$v_{1,1}(x,t) = \frac{(x-\pi)\,(x-1)}{12\,e\,(\pi-1)} \cos(y) \Big((e^2-1)\,(2\pi - x - 1) + + 2\,e\,(x - 2 + \pi) \sinh(\pi)\Big),$$
$$v_{2,1}(x,t) = \Big(\frac{\pi\,y}{6} - \frac{y^2}{2} + \frac{y^3}{3\,\pi}\Big) \sinh(x).$$

Table 1
Error in the temperature reconstruction (Δ_u – absolute error, δ_u – relative error)

n	Δ_u	δ_u [%]
0	0.47058	12.0836
1	0.37189	9.5494
2	0.12477	3.2038
3	0.06857	1.7607
4	0.02902	0.7451
5	0.01417	0.3639
6	0.00641	0.1646

In Table 1, the errors in reconstruction of the function describing distribution of temperature in considered region are presented. Displayed results show that the errors rapidly decrease together with the increasing number of components in sum (8).

3. Unsteady state heat conduction

Now will we discuss an application of the homotopy perturbation method for solving the unsteady state heat conduction problem described by means of the equation

$$\frac{\partial u}{\partial t}(x,t) = a\,\frac{\partial^2 u(x,t)}{\partial x^2}, \qquad (x,t) \in D, \tag{24}$$

where a denotes the thermal diffusivity and $D = \{(x,t);\ x \in (b_1, b_2),\ t \in (0, t^*)\}$. The initial condition is also given

$$u(x,0) = \psi(x), \qquad x \in [b_1, b_2], \tag{25}$$

as well as the boundary conditions of the first kind

$$u(b_1,t) = \varphi_1(t), \qquad t \in (0,t^*), \tag{26}$$

$$u(b_2,t) = \varphi_2(t), \qquad t \in (0,t^*). \tag{27}$$

We start by defining the homotopy operator for equation (24):

$$H(v,p) := \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} + p\,\Big(\frac{\partial^2 u_0}{\partial x^2} - \frac{1}{a}\,\frac{\partial v}{\partial t}\Big). \tag{28}$$

Solution of equation $H(v,p) = 0$ will be sought in form of the series

$$v = \sum_{j=0}^{\infty} p^j\, v_j. \tag{29}$$

Proceeding similarly as in the previous case, we receive $v_0 = u_0$ together with the following partial differential equations

$$\frac{\partial^2 v_1}{\partial x^2} = \frac{1}{a}\,\frac{\partial v_0}{\partial t} - \frac{\partial^2 u_0}{\partial x^2}, \tag{30}$$

and for $j \geqslant 2$:

$$\frac{\partial^2 v_j}{\partial x^2} = \frac{1}{a}\,\frac{\partial v_{j-1}}{\partial t}. \tag{31}$$

For the first of the above equations we define the conditions

$$\begin{cases} v_0(b_1,t) + v_1(b_1,t) = \varphi_1(t), \\ v_0(b_2,t) + v_1(b_2,t) = \varphi_2(t), \end{cases} \tag{32}$$

whereas, for the second equation we define conditions of the form ($j \geqslant 2$):

$$\begin{cases} v_j(b_1,t) = 0, \\ v_j(b_2,t) = 0. \end{cases} \tag{33}$$

As the initial approximation u_0 we can take the function describing the initial condition

$$u_0(x,t) = \psi(x). \tag{34}$$

Example 3.1

We will illustrate an application of the proposed method by the example in which: $b_1 = 0$, $b_2 = 1$, $a = 1$, $t^* = 2$ and:

$$\begin{aligned}
\psi(x) &= \frac{1}{24}\,x^4,\\
\varphi_1(t) &= \frac{1}{2}\,t^2,\\
\varphi_2(t) &= \frac{1}{24} + \frac{1}{2}\,t + \frac{1}{2}\,t^2.
\end{aligned}$$

As the initial approximation u_0 we take the function satisfying the initial condition, thus

$$v_0(x,t) = u_0(x,t) = \frac{1}{24}\,x^4.$$

By solving equation (30) with the boundary conditions (32) we find

$$v_1(x,t) = \frac{t^2}{2} + \Big(\frac{1}{24} + \frac{t}{2}\Big)\,x - \frac{x^4}{24}.$$

The successive functions $v_j(x,t)$, $j \geqslant 2$, are determined by solving equations (31) with the conditions (33). We obtain

$$\begin{aligned}
v_2(x,t) &= \Big(-\frac{1}{12} - \frac{t}{2}\Big)\,x + \frac{t\,x^2}{2} + \frac{x^3}{12},\\
v_3(x,t) &= \frac{x}{24} - \frac{x^3}{12} + \frac{x^4}{24},
\end{aligned}$$

and

$$v_j(x,t) = 0, \qquad j \geqslant 4.$$

In this way, we find the exact distribution of temperature in the entire considered region

$$u(x,t) = \sum_{j=0}^{\infty} v_j(x,t) = \frac{1}{24}\,x^4 + \frac{1}{2}\,t\,x^2 + \frac{1}{2}\,t^2.$$

Example 3.2

In the next example we assume $b_1 = 0$, $b_2 = 1$, $a = \frac{5}{2}$, $t^* = 1$ and

$$\begin{aligned}\psi(x) &= e^{(3-2x)/10},\\ \varphi_1(t) &= e^{(t+3)/10},\\ \varphi_2(t) &= e^{(t+1)/10}.\end{aligned}$$

Exact solution of the above formulated problem is of the form [14]:

$$u(x,t) = e^{(t-2x+3)/10}.$$

As the initial approximation u_0 we take the function satisfying the initial condition, it means

$$v_0(x,t) = u_0(x,t) = e^{(3-2x)/10}.$$

Table 2

Error in the temperature reconstruction (Δ_u – absolute error, δ_u – relative error)

n	Δ_u	δ_u [%]
1	$4.69735 \cdot 10^{-3}$	0.36431
2	$1.90225 \cdot 10^{-5}$	$1.47532 \cdot 10^{-3}$
3	$7.70944 \cdot 10^{-8}$	$5.97918 \cdot 10^{-6}$
4	$3.12452 \cdot 10^{-10}$	$2.42327 \cdot 10^{-8}$
5	$1.26633 \cdot 10^{-12}$	$9.82126 \cdot 10^{-11}$
6	$5.11722 \cdot 10^{-15}$	$3.96874 \cdot 10^{-13}$
7	$9.14762 \cdot 10^{-17}$	$7.09458 \cdot 10^{-15}$

By solving the appropriate equations we receive

$$\begin{aligned}v_1(x,t) &= -e^{(3-2x)/10} + e^{(t+3)/10}\,(1-x) + e^{(t+1)/10}\,x,\\ v_2(x,t) &= e^{(t+3)/10}\Big(-\frac{x}{75} + \frac{x^2}{50} - \frac{x^3}{150}\Big) + e^{(t+1)/10}\Big(-\frac{x}{150} + \frac{x^3}{150}\Big),\\ v_3(x,t) &= e^{(t+3)/10}\Big(\frac{x}{28125} - \frac{x^3}{11250} + \frac{x^4}{15000} - \frac{x^5}{75000}\Big) +\\ &\quad + e^{(t+1)/10}\Big(\frac{7\,x}{225000} - \frac{x^3}{22500} + \frac{x^5}{75000}\Big).\end{aligned}$$

In Table 2 the errors in reconstruction of the function describing distribution of temperature in considered region are displayed. Presented results indicate that the errors rapidly decrease together with the increasing number of components. Error of satisfying the initial condition for the 2− and 5−order approximations are showed in Figures 1 and 2, respectively. For the 7−order approximation the error of satisfying the initial condition does not exceed the value $2.5 \cdot 10^{-16}$. Whereas, the boundary conditions for $x = b_1$ and $x = b_2$ are fulfilled precisely which is the consequence of the proper selection of boundary conditions for equations (30) and (31).

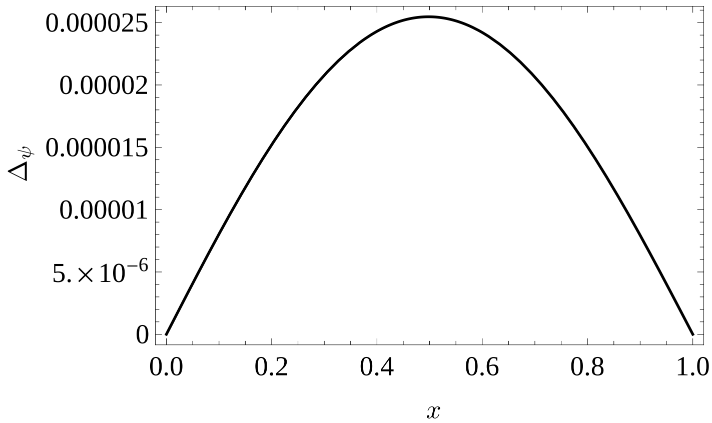

Fig. 1. Error in reconstruction of the initial condition for 2-order approximate solution
Rys. 1. Błąd spełnienia warunku początkowego dla przybliżenia drugiego rzędu

4. Conclusions

By applying the homotopy perturbation method we receive the function series convergent to the solution of considered problem (under the proper assumptions). In many cases it is possible to determine the sum of the obtained series, which means, to calculate the exact solution of the problem. In those cases in which determining the sum of series is impossible, we can use the initial components of the series and form the approximate solution. With regard to the rapid convergent of considered series, just few initial components assure vary small error of approximate solution.

The great advantage of applied method is that it does not require discretization of the region, like in the case of classical methods based on the finite-difference

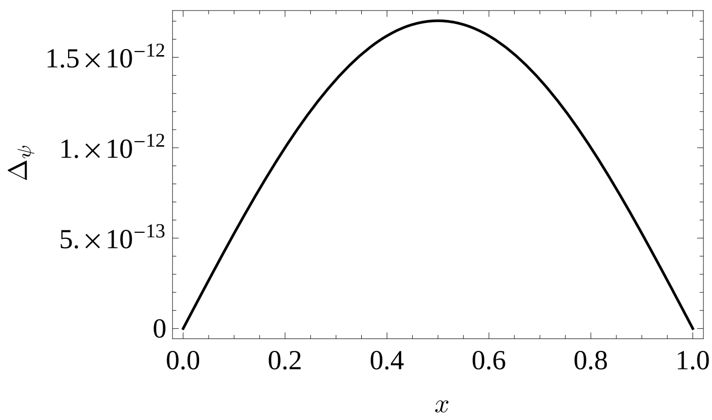

Fig. 2. Error in reconstruction of the initial condition for 5-order approximate solution
Rys. 2. Błąd spełnienia warunku początkowego dla przybliżenia piątego rzędu

method or the finite-element method. The proposed method produces a wholly satisfactory result already in a small number of iterations, whereas the classical methods require a suitably dense mesh in order to achieve similar accuracy.

References

1. Abbasbandy S.: *Homotopy analysis method for heat radiation equations.* Int. Comm. Heat & Mass Transf. **34** (2007), 380–387.
2. Biazar J., Ghazvini H.: *Convergence of the homotopy perturbation method for partial differential equations.* Nonlinear Anal.: Real World Appl. **10** (2009), 2633–2640.
3. Ganji D.D., Rajabi A.: *Assessment of homotopy-perturbation and perturbation methods in heat radiation equations.* Int. Comm. Heat & Mass Transf. **33** (2006), 391–400.
4. Grzymkowski R.: *Nieklasyczne metody rozwiązywania zagadnień przewodzenia ciepła.* Wyd. Pol. Śl., Gliwice 2010.
5. Grzymkowski R., Hetmaniok E., Słota D.: *Application of the homotopy perturbation method for calculation of the temperature distribution in the cast-mould heterogeneous domain.* J. Achiev. Mater. Manuf. Eng. **43** (2010), 299–309.
6. He J.-H.: *Homotopy perturbation technique.* Comput. Methods Appl. Mech. Engrg. **178** (1999), 257–262.

7. He J.-H.: *Comparision of homotopy perturbation method and homotopym analysis method.* Appl. Math. Comput. **156** (2004), 527–539.
8. He J.-H.: *Non-Perturbative Methods for Strongly Nonlinear Problems.* Dissertation.de-Verlag, Berlin 2006.
9. Khan Y., Wu Q.: *Homotopy perturbation transform method for nonlinear equations using Hes polynomials*, Comput. Math. Appl. **61** (2011), 1963–1967.
10. Liao S.J.: *Beyond Perturbation: Introduction to the Homotopy Analysis Method.* Chapman and Hall–CRC Press, Boca Raton 2003.
11. Madani M., Fathizadeh M., Khan Y., Yildirim A.: *On the coupling of the homotopy perturbation method and Laplace transformation*, Math. Comput. Modelling **53** (2011), 1937–1945.
12. Nayfeh A.H.: *Perturbation Method.* Wiley, New York 1973.
13. Słota D.: *The application of the homotopy perturbation method to one-phase inverse Stefan problem.* Int. Comm. Heat & Mass Transf. **37** (2010), 587–592.
14. Słota D.: *Homotopy perturbation method for solving the two-phase inverse Stefan problem.* Numer. Heat Transfer A **59** (2011), 755–768.

Omówienie

W artykule przedstawiono zastosowanie homotopijnej metody perturbacyjnej do rozwiązania ustalonego zagadnienia przewodzenia ciepła, opisanego równaniem Laplace'a. Przedstawiono także sposób wykorzystania omawianej metody do rozwiązania zagadnienia nieustalonego przewodzenia ciepła. Zaprezentowane zastosowania zilustrowane zostały przykładami.

Stosując homotopijną metodę perturbacyjną otrzymujemy szereg funkcyjny, który jest zbieżny do rozwiązania rozważanego zagadnienia (przy odpowiednich założeniach). W wielu przypadkach można wyznaczyć sumę uzyskanego szeregu, a tym samym otrzymać dokładne rozwiązanie rozważanego zagadnienia. W przypadkach gdy nie jesteśmy w stanie wyznaczyć analitycznie sumy szeregu do budowy rozwiązania przybliżonego możemy wykorzystać jego początkowe składniki. Ze względu na szybką zbieżność otrzymanego szeregu, już kilka początkowych wyrazów zapewnia bardzo mały błąd odtworzenia rozwiązania dokładnego.