ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 2011

Seria: MATEMATYKA STOSOWANA z. 1 Nr kol. 1854

Roman WITUŁA, Natalia GAWROŃSKA, Damian SŁOTA

Institute of Mathematics
Silesian University of Technology

BASIC PROPERTIES OF THE FULL MATRICES

Summary. In this paper the, so called, full matrices are distinguished. A number of basic properties of such matrices are also presented. Moreover, few possible directions for further research are indicated.

PODSTAWOWE WŁASNOŚCI MACIERZY PEŁNYCH

Streszczenie. W artykule wyróżniono tzw. macierze pełne. Przedstawiono wiele podstawowych własności tych macierzy. Wskazano też kilka możliwych kierunków badań.

Aim of this paper is to determine some basic properties of the full matrices. Let us begin with definition of the discussed matrices.

Definition 1. *Matrix $A = [a_{ij}] \in \mathbb{M}_{n\times m}(\mathbb{C})$ is said to be the full matrix if all its elements are different from zero ($a_{ij} \neq 0$).*

2010 Mathematics Subject Classification: 15B99, 15B36, 15A23.
Wpłynęło do Redakcji (received): 22.06.2011 r.
Natalia Gawrońska is a student of the Faculty of Applied Mathematics in Silesian University of Technology.

One should emphasize the fact that the full matrices (or the "almost" full matrices, it means, matrices with the small number of zero elements in comparison with the non-zero elements), together with the sparse matrices, play rather important role in technics. Among others, they appear in the finite element method, computer graphics, data compression, filtration and optics [3, 4]. Moreover, they can be used for testing various algorithms and this specific application was the main cause of our interests in the full matrices.

One can easy give the examples of singular full square matrices, as well as of non-singular full square matrices, of any order $n \in \mathbb{N}$.

Matrix $[1]_{n\times n}$ is singular and full. Let us consider the matrices

$$A_n = \begin{bmatrix} 1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{1} & & 1 \end{bmatrix}_{n\times n}, \qquad B_n = \begin{bmatrix} 1 & & \mathbf{1} \\ & \ddots & \\ \mathbf{0} & & 1 \end{bmatrix}_{n\times n}.$$

Matrices A_n and B_n are obviously invertible, thus their product matrix $A_n\,B_n$ is also the invertible matrix. But,

$$A_n\,B_n = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 2 & 2 & \dots & 2 & 2 \\ 1 & 2 & 3 & \dots & 3 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & \dots & n-1 & n-1 \\ 1 & 2 & 3 & \dots & n-1 & n \end{bmatrix}_{n\times n},$$

so, $A_n\,B_n$ is the full matrix as well.

Theorem 2. *For each $n \in \mathbb{N}$ there exists a full matrix A_n with coefficients in $\mathbb{N}$ and determinant equal to one.*

Proof. For $n = 1$ we have $A_1 = [1]$, for $n = 2$ we have, for example

$$A_2 = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}.$$

The other matrices A_2 can be obtained by using the known fact from the elementary number theory [1]: if $r, s \in \mathbb{N}$, $(r, s) = 1$, then there exist $p, q \in \mathbb{N}$ such that $p\,r - q\,s = 1$.

In general, let us assume that for some $n \in \mathbb{N}$ a matrix $A_n = [\alpha_{ij}]_{n\times n}$ is given, such that it satisfies the thesis of theorem. Then we put

$$A_{n+1} = \begin{bmatrix} 2 & 1 & \dots & 1 \\ \alpha_{11} & & & \\ \vdots & & A_n & \\ \alpha_{n1} & & & \end{bmatrix}.$$

We can easily verify (by expanding with respect to the first row) that

$$\det A_{n+1} = 2 \cdot \det A_n - 1 \cdot \det A_n = 1.$$

Indeed, it is enough to examine the expanded form of the matrix A_{n+1}:

$$A_{n+1} = \begin{bmatrix} 2 & 1 & 1 & \dots & 1 \\ \alpha_{11} & \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{bmatrix}.$$

□

Let us notice that, in the above proof, in definition of the first row of matrix A_{n+1} only taking 2 as the first element and 1 as the second element is of great importance. Other elements in this row can be any numbers. Substituting in this row number 2 by $s \in \mathbb{N}$ we receive the matrix $A_{n+1}(s)$ possessing determinant equal to $s - 1$. Thus, the following conclusion results.

Corollary 3. *For any $s \in \mathbb{N} \cup \{0\}$ and any $n \in \mathbb{N}$ there exists a full matrix $A \in \mathbb{M}_{n\times n}(\mathbb{N})$ such that* $\det A = s$.

Theorem 4. *For each $n \in \mathbb{N}$ there exists a full matrix A_n with different coefficients in $\mathbb{N}$ and of determinant equal to one.*

Proof. For $n = 1$ we have $A_1 = [1]$, whereas for $n = 2$ we have

$$A_2 = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}.$$

Now, let us suppose that for some $n \in \mathbb{N}$, $n \geqslant 2$, we have determined the full matrix $A_n = [\alpha_{ij}]_{n\times n}$ with different coefficients in $\mathbb{N}$, such that $\det A_n = 1$. Then, the matrix A_{n+1} is defined in the following way

$$A_{n+1} = \begin{bmatrix} \beta_1 & \beta_2 & \dots & \beta_{n+1} \\ M\,\alpha_{11} & & & \\ \vdots & & A_n & \\ M\,\alpha_{n1} & & & \end{bmatrix},$$

where

$$\begin{aligned} M &:= 1 + \max\{\alpha_{ij} :\ 1 \leqslant i, j \leqslant n\}, \\ \beta_2 &:= 1 + \max\{M\,\alpha_{i1} :\ 1 \leqslant i \leqslant n\}, \\ \beta_1 &:= M\,\beta_2 + 1, \end{aligned}$$

whereas

$$\beta_3 := \beta_1 + 1, \quad \beta_4 := \beta_3 + 1, \quad \dots, \quad \beta_{n+1} := \beta_n + 1.$$

□

Remark 5. There exist examples of full matrices with coefficients in $\mathbb{N}$, inverse matrices of which are the full matrices as well. For example the circulant matrices with Fibonacci and Lucas numbers [2] $\mathrm{Circ}(F_1, \dots, F_n)$ and $\mathrm{Circ}(L_1, \dots, L_n)$, respectively, possesses this property for every $n \in \mathbb{N}$. Here

$$\mathrm{Circ}(a_1, a_2, \dots, a_n) = \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_n & a_1 & \dots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_3 & a_4 & \dots & a_n & a_1 \end{bmatrix}.$$

Proofs of these facts are not elementary.

We can easily verify that the inverse matrix of the invertible full 2×2 matrix is the full matrix as well. Indeed the following formula holds

$$\begin{bmatrix} p & q \\ s & t \end{bmatrix}^{-1} = \frac{1}{p\,t - q\,s} \begin{bmatrix} t & -q \\ -s & p \end{bmatrix}.$$

Theorem 6. *For every $n \in \mathbb{N}$, $n > 1$, there exists a full matrix B_n with irrational coefficients and determinant equal to one.*

Proof. Let $A_n = [\alpha_{ij}]_{n \times n}$, $n \in \mathbb{N}$, be the matrices from Theorem 2 We assume

$$B_2 = \begin{bmatrix} 3\,\pi & 2\,\pi \\ \frac{4}{\pi} & \frac{3}{\pi} \end{bmatrix},$$

$$B_{n+1} = \begin{bmatrix} 2\pi & \pi \;\; e & \dots & e \\ \dfrac{\alpha_{11}}{\pi} & & & \\ \vdots & & \dfrac{1}{\pi} A_n & \\ \dfrac{\alpha_{n1}}{\pi} & & & \end{bmatrix}, \qquad n = 2, 3, \dots .$$

□

If α, β are the irrational numbers linearly independent over $\mathbb{Q}$ and α^2, β^2, $\alpha^2\,\beta$, $\beta^2\,\alpha$ and $\alpha\,\beta$ are also irrational (in fact, we can formulate weaker assumptions about the numbers α and β, which results from the appropriate analysis of the matrix given below), then one can obtain the full matrix of dimensions 3×3 with different irrational coefficients and determinant equal to one, in the following way

$$\begin{bmatrix} 2\,\alpha & \alpha\,\beta & \dfrac{\beta}{3} \\ \dfrac{3}{\beta} & \dfrac{3}{\alpha} & \dfrac{1}{\alpha} \\ \dfrac{7}{\beta} & \dfrac{5}{\alpha} & \dfrac{2}{\alpha} \end{bmatrix}.$$

For example, we can select $\alpha = e$ and $\beta = e^2$. There exists the following generalization of this example.

Theorem 7. *For each $n \in \mathbb{N}$, $n > 1$, there exists a full matrix A_n with different irrational coefficients and determinant equal to one.*

Proof. First, we determine the sequence $\{\alpha_n\}$ of the irrational numbers, of the properties

$$\prod_{j=1}^{t} \alpha_j^{s_j} \notin \mathbb{Q}, \qquad \text{where } s_j \in \mathbb{Z},\ 1 \leqslant j \leqslant t,\ \sum |s_j| > 0,$$

for example $\alpha_n = e^n$. Next, we set

$$A_2 = \begin{bmatrix} 7\,\alpha_1 & 4\,\alpha_2 \\ \dfrac{5}{\alpha_2} & \dfrac{3}{\alpha_1} \end{bmatrix}.$$

In general, by having the matrix A_n we can define the matrix A_{n+1} in the following manner

$$A_{n+1} = \begin{bmatrix} \alpha_{n+1}^2 & \alpha_{n+1} & \alpha_{n+1}^3 & \cdots & \alpha_{n+1}^{n+1} \\ \alpha_{n+1}\,\alpha_{11} & & & & \\ \vdots & & & A_n & \\ \alpha_{n+1}\,\alpha_{n1} & & & & \end{bmatrix}.$$

□

Theorem 8. *For every $n \in \mathbb{N}$, there exists a full matrix $A_n \in \mathbb{M}_{n\times n}(\{1,2\})$, such that* $\det A_n = 1$.

Proof. If $n \geqslant 2$, then it is sufficient to take

$$A_n = \begin{bmatrix} 2 & & & \mathbf{1} \\ & \ddots & & \\ & & 2 & \\ \mathbf{1} & & & 1 \end{bmatrix}.$$

□

Let $\mathcal{P}_n \subset \mathbb{M}_{n\times n}(\mathbb{N})$, where $n \in \mathbb{N}$, $n > 1$, be the family of all full matrices of determinant equal to one. One can easily notice that the families $\mathcal{P}_n$ are the semigroups with respect to the multiplication of matrices. Element A of family $\mathcal{P}_n$ is called to be prime if it is not a product of any two elements from $\mathcal{P}_n$. The following theorem holds.

Theorem 9. *The set of generators of family $\mathcal{P}_n$ is infinite.*

Proof. It is enough to show that for every $n \in \mathbb{N}$, $n > 1$, the set of prime elements of family $\mathcal{P}_n$ is infinite. For example, for $\mathcal{P}_2$ each matrix of the form

$$\begin{bmatrix} m\,n+1 & m \\ n & 1 \end{bmatrix}, \qquad m, n \in \mathbb{N},$$

is prime. This is the consequence of the simple observation that if one of elements of the matrix $A \in \mathcal{P}_n$ is equal to one, then the matrix A is prime.

In $\mathcal{P}_3$, for example the matrices of the form given below are prime

$$\begin{bmatrix} 2 & 1 & 1 \\ mn+1 & mn+1 & m \\ n & n & 1 \end{bmatrix}, \qquad m, n \in \mathbb{N},$$

whereas, in $\mathcal{P}_4$ the following matrices are prime

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ mn+1 & mn+1 & mn+1 & m \\ n & n & n & 1 \end{bmatrix}, \qquad m, n \in \mathbb{N}, \quad \text{etc.}$$

$\square$

Problem 1. *With every matrix $A = [\alpha_{ij}]_{n\times n} \in \mathcal{P}_n$ we will associate the number*

$$s(A) := \sum_{i,j=1}^{n} \alpha_{ij}.$$

Is it true that the matrices from the proof of Theorem 9 realize $\min\{s(A) : A \in \mathcal{P}_n\}$ *?*

Now, we will discuss the next, important for numerical applications, problem. Is it true that each matrix $A \in \mathbb{M}_{n\times n}(\mathbb{R})$ is a product of two full matrices?

Theorem 10. *Each full matrix $A = [\alpha_{ij}] \in \mathbb{M}_{n\times n}(\mathbb{R})$ can be presented as a product of two full matrices both with real coefficients.*

Proof. Let us denote

$$\mathbf{M} := \begin{bmatrix} M & & \mathbf{1} \\ & \ddots & \\ \mathbf{1} & & M \end{bmatrix}.$$

We seek the full matrix $B = [\beta_{ij}] \in \mathbb{M}_{n\times n}(\mathbb{R})$ such that

$$\mathbf{M}\, B = A,$$

or, equivalently

$$\mathbf{M} \begin{bmatrix} \beta_{1j} \\ \beta_{2j} \\ \vdots \\ \beta_{nj} \end{bmatrix} = \begin{bmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{nj} \end{bmatrix}, \qquad j = 1, 2, \ldots, n,$$

which, on the basis of the Cramer's rules, for sufficiently large M is equivalent to the following equalities

$$\beta_{ij} = \frac{1}{\det \mathbf{M}} \det \underbrace{\begin{bmatrix} M & 1 & \dots & 1 & \alpha_{1j} & 1 & \dots & 1 \\ 1 & M & \dots & 1 & \alpha_{2j} & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & \dots & 1 & \alpha_{nj} & 1 & \dots & M \end{bmatrix}}_{i\text{-th column}} \qquad 1 \leqslant i, j \leqslant n.$$

Both determinants in this formula are non-zero, since $\det \mathbf{M} = M^n + \dots$ and

$$\det \begin{bmatrix} M & 1 & \dots & 1 & \alpha_{1j} & 1 & \dots & 1 \\ 1 & M & \dots & 1 & \alpha_{2j} & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & \dots & 1 & \alpha_{nj} & 1 & \dots & M \end{bmatrix} = \alpha_{ij}\, M^{n-1} + \dots .$$

□

Theorem 11. *For each full matrix $B \in \mathbb{M}_{n\times n}(\mathbb{R})$ of the order smaller than n and for each $k \in \{1, 2, \dots, n-1\}$ there exists a full matrix $A \in \mathbb{M}_{n\times n}(\mathbb{R})$ of order k such that*

$$A\, B = \mathbb{O},$$

where $\mathbb{O}$ denotes the zero matrix.

Proof of this theorem (of technical nature) will be omitted in this paper (mostly because of its capacity).

Theorem 12. *A matrix $A \in \mathbb{M}_{2\times 2}(\mathbb{R})$ can be presented as a product of two full matrices if and only if it is not of the form*

$$\begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix}, \qquad \alpha \neq 0,$$

with the accuracy of its elements permutations.

Proof. Let us set

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathbb{M}_{2\times 2}(\mathbb{R}).$$

On can easily notice that if

$$\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

then

$$x = \det \begin{bmatrix} \alpha & 5 \\ \gamma & 2 \end{bmatrix}, \qquad y = \det \begin{bmatrix} 3 & \alpha \\ 1 & \gamma \end{bmatrix},$$
$$z = \det \begin{bmatrix} \beta & 5 \\ \delta & 2 \end{bmatrix}, \qquad w = \det \begin{bmatrix} 3 & \beta \\ 1 & \delta \end{bmatrix},$$

moreover, the matrix $\begin{bmatrix} x & z \\ y & w \end{bmatrix}$ is full matrix if the following conditions hold $\alpha \neq \frac{5}{2}\,\gamma$, $\alpha \neq 3\,\gamma$, $\beta \neq \frac{5}{2}\,\delta$ i $\beta \neq 3\,\delta$.

We assume that in the matrix

$$\begin{bmatrix} x & z \\ y & w \end{bmatrix}$$

exactly one element is equal to zero.

Let us suppose that, for example, $\alpha = \frac{5}{2}\,\gamma$ and $\alpha\,\beta\,\gamma \neq 0$. Then we select the sufficiently large prime numbers p, q and the numbers $n, m \in \mathbb{N}$, such that

$$p\,n - q\,m = 1.$$

One can easily verify that there exists then the full matrix $\begin{bmatrix} x & z \\ y & w \end{bmatrix}$, such that

$$\begin{bmatrix} p & q \\ m & n \end{bmatrix} \begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

which results from the following fact

$$\det \begin{bmatrix} \alpha & q \\ \gamma & n \end{bmatrix} \neq 0 \Leftrightarrow \alpha \neq \frac{q}{n}\,\gamma \Leftrightarrow 2\,q \neq 5\,n,$$

$$\det \begin{bmatrix} \alpha & p \\ \gamma & m \end{bmatrix} \neq 0 \Leftrightarrow \alpha \neq \frac{p}{m}\,\gamma \Leftrightarrow 5\,m \neq 2\,p,$$

similarly

$$\det \begin{bmatrix} \beta & q \\ \delta & n \end{bmatrix} \neq 0 \Leftrightarrow \beta \neq \frac{q}{n}\,\delta,$$

and

$$\det \begin{bmatrix} \beta & p \\ \delta & m \end{bmatrix} \neq 0 \Leftrightarrow \beta \neq \frac{p}{m}\,\delta.$$

In case, in which the given matrix is of the form

$$\begin{bmatrix} 0 & \alpha \\ 0 & \beta \end{bmatrix}, \qquad \alpha\,\beta \neq 0,$$

we have

$$\begin{bmatrix} \frac{\alpha}{2} & \frac{\alpha}{2} \\ \frac{\beta}{2} & \frac{\beta}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ 0 & \beta \end{bmatrix}.$$

The last thing, we need to prove, is that each matrix of the form

$$\begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix}, \qquad \alpha \neq 0,$$

cannot be written as a product of two full matrices.

Let as assume the opposite fact. Let

$$\begin{bmatrix} x & z \\ y & w \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix}.$$

Then we have

$$\frac{x}{z} = -\frac{r}{p} \qquad \text{and} \qquad \frac{x}{z} = -\frac{s}{q},$$

which implies

$$\frac{r}{p} = \frac{s}{q}.$$

Moreover, we have

$$\frac{y}{w} = -\frac{r}{p}.$$

Hence

$$\alpha = w\,q\left(\frac{y}{w} + \frac{s}{q}\right) = w\,q\left(-\frac{s}{q} + \frac{s}{q}\right) = 0,$$

which is impossible. □

Remark 13. Matrices of the form

$$\begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix}, \qquad \alpha \neq 0,$$

can be presented as a product of three full matrices

$$\begin{bmatrix} 1 & -1 \\ \frac{\alpha}{2} & \frac{\alpha}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ \frac{\alpha}{2} & \frac{\alpha}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix}.$$

Trials for generalizing Theorem 12 for matrices of higher order are in progress. We will present in this paper only the sample of our results.

Theorem 14. *Let $A \in \mathbb{M}_{3\times 3}(\mathbb{R})$ be the full matrix. If the two first rows of the matrix A are linearly independent and if for the given row $W = [\alpha, \beta, \gamma] \in \mathbb{R}^3$ there exists a full matrix $B \in \mathbb{M}_{3\times 3}(\mathbb{R})$ such that*

$$A\,B = \begin{bmatrix} \mathbb{O} \\ \mathbb{O} \\ W \end{bmatrix}, \tag{1}$$

then either $\alpha\,\beta\,\gamma \neq 0$ or $\alpha = \beta = \gamma = 0$.

Proof. Let W_1, W_2, W_3 be the successive rows of the matrix B and let $A = [a_{ij}]_{3\times 3}$. Then, the equality (1) can be written in in the following way

$$\begin{cases} a_{11}\,W_1 + a_{12}\,W_2 + a_{13}\,W_3 = \mathbb{O}, \\ a_{21}\,W_1 + a_{22}\,W_2 + a_{23}\,W_3 = \mathbb{O}, \\ a_{31}\,W_1 + a_{32}\,W_2 + a_{33}\,W_3 = W, \end{cases} \tag{2}$$

from where we get

$$W_3 = -\frac{a_{11}}{a_{13}}\,W_1 - \frac{a_{12}}{a_{13}}\,W_2 = -\frac{a_{21}}{a_{23}}\,W_1 - \frac{a_{22}}{a_{23}}\,W_2, \tag{3}$$

which implies the equality

$$b\,W_1 = c\,W_2, \tag{4}$$

for some $b, c \in \mathbb{R}$. If $b = 0$, then $c = 0$ as well, and vice versa, since the rows W_1 and W_2 are full. However, the equality $b = c = 0$ means that

$$\frac{a_{11}}{a_{21}} = \frac{a_{13}}{a_{23}} = \frac{a_{12}}{a_{22}},$$

that is the linear dependence of the two first rows of matrix A should take place which is contrary to the assumption.

Therefore, $b\,c \neq 0$ and from (4), (3) and the last equation of the system (2) it results that $d\,W_1 = W$, for some $d \in \mathbb{R}$, which is possible only if $W = \mathbb{O}$ or W is full. $\square$

The above theorem can be generalized in the following manner.

Theorem 15. *Let $A \in \mathbb{M}_{n\times n}(\mathbb{R})$, $n \geqslant 3$, be the full matrix. If $n-1$ first rows of the matrix A are linearly independent and if for the given row $W = [w_1, w_2, \ldots, w_n] \in \mathbb{R}^n$ there exists a full matrix $B \in \mathbb{M}_{n\times n}(\mathbb{R})$ such that*

$$A\,B = \begin{bmatrix} \mathbb{O} \\ \vdots \\ \mathbb{O} \\ W \end{bmatrix}, \tag{5}$$

then either $w_1\, w_2\, \ldots\, w_n \neq 0$ or $w_1 = w_2 = \ldots = w_n = 0$.

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Omówienie

W artykule wyróżniono tzw. macierze pełne. Przedstawiono wiele podstawowych własności tych macierzy. Podano także rekurencyjne metody generowania macierzy kwadratowych pełnych o elementach wymiernych oraz niewymiernych i wyznaczniku równym jeden. Omawiany jest problem rozkładu dowolnej macierzy kwadratowej na iloczyn dwóch macierzy pełnych. Dla macierzy o wymiarze 2×2 problem ten został całkowicie rozstrzygnięty. Wyróżniono też rodzinę $\mathcal{P}_n \subset \mathbb{M}_{n\times n}(\mathbb{N})$ macierzy pełnych o wyznaczniku równym jeden. Oczywiście $\mathcal{P}_n$ jest dla każdego $n \in \mathbb{N}$ półgrupą nieprzemienną bez jedynki ze względu na mnożenie macierzy. Udowodniono, że $\mathcal{P}_n$ posiada nieskończony zbiór elementów pierwszych. Wskazano też kilka możliwych kierunków badań.