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SHARPLY-WEAK REGULAR LINEAR EXTENSIONS OF DYNAMICAL SYSTEMS LEADING UP TO REGULAR SYSTEMS

Summary. In this paper we consider some sign–changing Lyapunov function in research on regularity and sharply-week regularity of sets of linear extensions of dynamical systems. By regularity we mean exponential dichotomy of linear differential systems.

DOPEŁNIENIE SŁABO REGULARNYCH LINIOWYCH ROZSZERZEŃ UKŁADÓW DYNAMICZNYCH DO UKŁADÓW REGULARNYCH

Streszczenie. W artykule podjęto tematykę regularności liniowych rozszerzeń układów dynamicznych, która równoważna jest z wykładniczą dychotomią liniowych układów różniczkowych. Przeprowadzono badanie regularności przy użyciu znakozmiennych funkcji Lapunowa. Ponadto, przedstawiono doprowadzenie słabo regularnych układów do regularnych.

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1. Introduction

Let us consider a system of differential equations

$$\frac{dx}{dt} = f(x), \quad \frac{dy}{dt} = A(x)y, \tag{1}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $f(x) = (f_1(x), \dots, f_m(x))$ – a vector function defined for all $x \in \mathbb{R}^m$, which satisfies the Lipschitz inequality locally. We use $C_{Lip}(\mathbb{R}^m)$ to stand for the space of functions $f(x) \in C^0(T_m)$. Moreover the Cauchy problem $\frac{dx}{dt} = f(x), x\big|_{t=0} = x_0$, has a solution $x = x(t; x_o)$ for every fixed $x_0 \in \mathbb{R}^m$ and the solution is defined for every $t \in \mathbb{R}$. Is is equivalent to $\|f(x)\| \leqslant \alpha_1\|x\| + \alpha_2$, for all $x \in \mathbb{R}^m$ with any positive constants α_1, α_2. Elements of the $n \times n$ – dimensional matrix $A(x)$ are real scalar functions which are continuous and bounded in $\mathbb{R}^m$.

Let us use notation: $C^0(\mathbb{R}^m)$ – a space of real continuous and bounded in $\mathbb{R}^m$ functions; $\langle y, \tilde{y}\rangle = \sum_{i=1}^{n} y_i\tilde{y}_i$ – an inner product in $\mathbb{R}^n$, $\|y\| = \sqrt{\langle y, y\rangle}$ – a square form $y \in \mathbb{R}^m$; $\Omega_\tau^t(x_0)$ – a fundamental matrix of the solutions of linear system

$$\frac{dy}{dt} = A(x(t; x_0))y,$$

which takes the value of I_n – n–dimensional identity matrix for $t = \tau$: $\Omega_\tau^t(x_0)\big|_{t=\tau} = I_n$; $C'(\mathbb{R}^m; f)$ – a subspace of $C^0(\mathbb{R}^m)$ of functions $F(x)$ such that superposition $F(x(t; x_0))$ is continuously differentiable with respect to t, where

$$\frac{d}{dt}F(x(t;x))\bigg|_{t=0} =^{df} \dot{F}(x) \in C^0(\mathbb{R}^m).$$

Index 0 in the Cauchy problem solution is frequently missed $x = x(t; x_0) = x(t; x)$.

Let us also consider system of differential equations

$$\frac{dx}{dt} = f(x), \quad \frac{dy}{dt} = A(x)y + h(x), \tag{2}$$

where the vector function $h(x) \in C^0(\mathbb{R}^m)$. Let us recall useful definitions [1].

Definition 1. *We say that the system (2) possesses an invariant bounded manifold determined by the equality*

$$y = u(x), \tag{3}$$

when $u(x) \in C'(\mathbb{R}^m; f)$ and the identity

$$\dot{u}(x) \equiv A(x)u(x) + h(x) \qquad \forall x \in \mathbb{R}^m$$

holds.

Definition 2. *Let $C(x)$ be an $(n \times n)$–dimensional continuous matrix, $C(x) \in C^0(T_m)$. Then the function $G_0(\tau; x)$:*

$$G_0(\tau, x) = \begin{cases} \Omega_\tau^0(x)C(x(\tau, x)), & \tau \leqslant 0, \\ \Omega_\tau^0(x)\left[C(x(\tau, x)) - I_n\right], & \tau > 0, \end{cases} \tag{4}$$

which fulfills the estimate

$$\|G_0(\tau, x)\| \leqslant Ke^{-\gamma|\tau|},$$

where K and γ are positive constants. The function (4) is called a Green function of an invariant bounded manifold for the system (1).

In case when the Green function (4) is unique the system (1) is called *regular*. When the system (1) possesses more then one Green function (4), then the system (1) is called *sharply–weak regular*.

When the Green function (4) exists then the invariant manifold (3) for the system (2) exists for every function $h(x) \in C^0(\mathbb{R}^m)$. The manifold can be defined by integral formula

$$y = u(x) = \int_{-\infty}^{\infty} G_0(\tau, x) \cdot h(x(\tau; x))d\tau.$$

Examples [2] exist in which the invariant bounded manifold (3) for the system (2) is unique for every function $h(x) \in C^0(\mathbb{R}^m)$ but the Green function (4) for the system (1) does not exist.

Researches of systems (1) with right sides defined on torus there are in [2–8]. In [2] the authors deal with problems of leading sharply–weak regular linear extensions of dynamical systems to regular systems.

It is obvious [2, 7, 8] that the system (1) is regular when the square form $V = \langle (S(x)y, y) \rangle$ with continuously differentiable nondegenerated symmetric matrix of coefficients $S(x) \in C'(\mathbb{R}_m; f)$ exists and its derivative along the solutions of the system (1) is positive or negative definite

$$\dot{V} = \left\langle \left[\frac{\partial S(x)}{\partial x} f(x) + S(x)A(x) + A^T(x)S(x) \right] y, y \right\rangle \geqslant \|y\|^2 \quad \forall y \in \mathbb{R}^n, \tag{5}$$

On the other hand the regularity of the system (1) means the existence of non-degenerated symmetric matrixes $S(x) \in C'(\mathbb{R}^m; f)$ which satisfy inequality (5).

Some of the matrixes can be defined by formula

$$S(x) = 2\int_{-\infty}^{0} \left[C(x) - I_n\right]^T \left[\Omega_0^z(x)\right]^T \Omega_0^z(x) \left[C(x) - I_n\right] dz - \\ - 2\int_{0}^{\infty} \left[C(x)\right]^T \left[\Omega_0^z(x)\right]^T \Omega_0^z(x) C(x) dz.$$

When the inequality (5) holds with symmetric matrix $S(x) \in C'(\mathbb{R}^m; f)$ and $\det S(\bar{x}) = 0$ for some $\bar{x} \in \mathbb{R}^m$, then the Green function (4) for the system (1) does not exist. Moreover many different Green function exist for the conjoint system to (1):

$$\frac{dx}{dt} = f(x), \quad \frac{dy_1}{dt} = -A^T(x) y_1, \quad y_1 \in \mathbb{R}^n. \tag{6}$$

In such systems researches of Green function and invariant manifolds dependence of parameters [7] is very difficult. Therefore let us lead up the system (6) to regular

$$\frac{dx}{dt} = f(x), \quad \frac{dy_1}{dt} = -A^T(x) y_1, \quad \frac{dy_2}{dt} = y_1 + A(x) y_2, \quad y_1, y_2 \in \mathbb{R}^n. \tag{7}$$

Furthermore the derivative along the solutions of the system (7) of nondegenerated square form

$$V_p = p \left\langle y_1, y_2 \right\rangle + \left\langle S(x) y_1, y_2 \right\rangle, \tag{8}$$

when the parameter $p \gg 0$, is positive defined. The method is effective in researches of Green function smoothness and invariant manifolds stability.

2. Main results

Les us apply the method to determine all solutions to the algebraic system

$$B(x) y = h(x), \tag{9}$$

where $B(x)$ – a rectangular matrix

$$B(x) = \begin{pmatrix} b_{11}(x) & b_{12}(x) & \dots & b_{1n}(x) \\ b_{21}(x) & b_{22}(x) & \dots & b_{2n}(x) \\ \dots & \dots & \dots & \dots \\ b_{p1}(x) & b_{p2}(x) & \dots & b_{pn}(x) \end{pmatrix},$$

which elements $b_{ij}(x)$ are real scalar functions, bounded and continous in $\mathbb{R}^m$. Rank of matrix $B(x)$ equals number of rows of the matrix

$$\operatorname{rank} B(x) \equiv p, \qquad p < n. \tag{10}$$

Certainly when the condition (10) is fulfilled the system (9) possesses many different solutions for every vector function $h(x)$. Our goal is to find the solutions forms. By analogy to (7) let us consider the system (9) which is leading up

$$\begin{cases} B(x)y = h(x), \\ y - B^T(x)z = q(x), \end{cases} \tag{11}$$

where $y \in \mathbb{R}^n$, $z \in \mathbb{R}^p$, $x \in \mathbb{R}^m$, $q(x)$ – a bounded and continuous in $\mathbb{R}^m$ vector function. Let us the second equation of the system (11):

$$y = B^T(x)z + q(x) \tag{12}$$

put in the system (9):

$$B(x)\left(B^T(x)z + q(x)\right) = h(x).$$

From the condition (10) follows that $\det B(x)B^T(x) \neq 0$, $\forall x \in \mathbb{R}^m$. Making sufficient calculations we will get z:

$$z = \left[B(x)B^T(x)\right]^{-1}\left[-B(x)q(x) + h(x)\right] =$$
$$= -\left[B(x)B^T(x)\right]^{-1} B(x)q(x) + \left[B(x)B^T(x)\right]^{-1} h(x).$$

Putting it to the solution (12), the form of all solutions to (9) is

$$y = \left\{I_n - B^T(x)\left[B(x)B^T(x)\right]^{-1} B(x)\right\} q(x) + B^T(x)\left[B(x)B^T(x)\right]^{-1} h(x). \tag{13}$$

Remark 3. In the solution (13) the matrix

$$I_n - B^T(x)\left[B(x)B^T(x)\right]^{-1} B(x) = P(x)$$

is the projection matrix

$$P^2(x) \equiv P(x), \qquad \forall x \in \mathbb{R}^m, \tag{14}$$

and the matrix $B^T(x)\left[B(x)B^T(x)\right]^{-1}$ is pseudoinverse to the matrix $B(x)$.

Remark 4. Every solution to the system (9) can be represented in the form (13) with certain vector function $q(x)$.

Example. Let us determine all solutions $y_j = y_j(x_1, x_2)$ to the linear scalar equation with periodic coefficients:

$$y_1 \cos x_1 \cos x_2 + y_2 \cos x_1 \sin x_2 + y_3 \sin x_1 = 0. \tag{15}$$

Solution. In this case the matrix $B(x)$ has the following form

$$B(x) = \begin{pmatrix} \cos x_1 \cos x_2, & \cos x_1 \cos x_2, & \sin x_1 \end{pmatrix}.$$

We are searching for a matrix equivalent to (13). Inasmuch as $B(x)B^T(x) \equiv 1$

$$I_n - B^T(x)\left[B(x)B^T(x)\right]^{-1} B(x) = I_n - B^T(x)B(x) =$$
$$= \begin{pmatrix} (1-\cos^2 x_1 \cos^2 x_2) & (-\cos^2 x_1 \cos x_2 \sin x_2) & (-\cos x_1 \sin x_1 \cos x_2) \\ (-\cos^2 x_1 \cos x_2 \sin x_2) & (1-\cos^2 x_1 \sin^2 x_2) & (-\cos x_1 \sin x_1 \sin x_2) \\ (-\cos x_1 \sin x_1 \cos x_2) & (-\cos x_1 \sin x_1 \sin x_2) & \cos^2 x_1 \end{pmatrix}.$$

All solutions to (14) are in the form

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} =$$
$$= \begin{pmatrix} (1-\cos^2 x_1 \cos^2 x_2) & (-\cos^2 x_1 \cos x_2 \sin x_2) & (-\cos x_1 \sin x_1 \cos x_2) \\ (-\cos^2 x_1 \cos x_2 \sin x_2) & (1-\cos^2 x_1 \sin^2 x_2) & (-\cos x_1 \sin x_1 \sin x_2) \\ (-\cos x_1 \sin x_1 \cos x_2) & (-\cos x_1 \sin x_1 \sin x_2) & \cos^2 x_1 \end{pmatrix}$$
$$\cdot \begin{pmatrix} q_1(x_1, x_2) \\ q_2(x_1, x_2) \\ q_3(x_1, x_2) \end{pmatrix},$$

where $q_1(x_1, x_2)$, $q_2(x_1, x_2)$, $q_3(x_1, x_2)$ – any real functions bounded in $\mathbb{R}^2$.

Let us come back to generalization of the symmetric system (7) in the form

$$\begin{gathered} \frac{dx}{dt} = f(x), \\ \frac{dy_1}{dt} = -A^T(x)y_1 + B_2(x)y_2, \quad \frac{dy_2}{dt} = B_1(x)y_1 + A(x)y_2, \quad y_1, y_2 \in \mathbb{R}^n, \end{gathered} \tag{16}$$

with the vector function $f(x)$ and the $n \times n$ – dimensional matrix $A(x)$, $B_j(x)$, $j = 1, 2$ which fulfill smoothness and boundedness conditions as in the system (1). The matrixes $B_j(x)$, $j = 1, 2$ are symmetric $B_j(x) \equiv B_j^T(x)$. The first matrix is positive definite

$$\langle B_1(x)y_1, y_1\rangle \geqslant \beta_1 \|y_1\|^2, \quad \beta_1 = \text{const} > 0, \tag{17}$$

and the second is nonnegative

$$\langle B_2(x)y_2, y_2\rangle \geqslant 0 \quad \forall y_1, y_2 \in \mathbb{R}^n. \tag{18}$$

When the conditions (5), (17), (18) hold the nondegenerated square form (8) derivative along the solutions to the system (7) is positive or negative definite for sufficiently big values of the parameter $p > 0$. Certainly in case when $B_2 \equiv 0$, $B_1 \equiv I_n$ the system (16) becomes the system (7).

Remark 5. When in the system (16) the matrixes $B_1(x), B_2(x) \in C^0(\mathbb{R}^m)$ fulfills the conditions (17), (18), and the condition (18) is weakened

$$\langle B_2(x)y_2, y_2\rangle \geqslant \beta_2\|y_2\|^2, \qquad \beta_2 = \text{const} > 0,$$

then the system (16) is regular for every matrix $A(x) \in C^0(\mathbb{R}^m)$.

It is easy to show that the square form $V = \langle y_1, y_2\rangle$, $y_j \in \mathbb{R}^n$ derivative along the solutions of the system (16) has the following form: $\dot{V} = \langle B_1 y_1, y_1\rangle + \langle B_2 y_2, y_2\rangle$. It is positive definite and it does not depend of the matrix $A(x) \in C^0(\mathbb{R}^m)$.

Let us notice the system (16) can be write down in the following form

$$\begin{gathered}\frac{dx}{dt} = f(x),\\ \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} \dfrac{dy_1}{dt} \\[2ex] \dfrac{dy_2}{dt} \end{pmatrix} = \\ = \left[\begin{pmatrix} B_1(x) & 0 \\ 0 & B_2(x) \end{pmatrix} + \begin{pmatrix} 0 & A(x) \\ -A^T(x) & 0 \end{pmatrix}\right] \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.\end{gathered} \tag{19}$$

In such a form the square form

$$V = \left\langle \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle$$

derivative along the solutions of the system (19) is

$$\dot{V} = 2\left\langle \begin{pmatrix} B_1(x) & 0 \\ 0 & B_2(x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle.$$

When the matrix $B(x) = \operatorname{diag}\{B_1(x), B_2(x)\}$ is positive or negative definite then the system (19) is regular for every matrix $A(x) \in C^0(\mathbb{R}^m)$. Let us generalize the system (19):

$$\frac{dx}{dt} = f(x), \quad S\frac{dy}{dt} = [B(x) + M(x)]\,y, \quad y \in \mathbb{R}^k, \tag{20}$$

where S – any nondegenerated symmetric matrix, $B(x)$ – a symmetric matrix, $M(x)$ – a skew-symmetric matrix $B(x), M(x) \in C^0(\mathbb{R}^m)$. In case in the system (20) number of parameters y is even $k = 2n$ and let us take a notation $y = (y_1, y_2)$, $y_j \in \mathbb{R}^n$ and the matrix S is in the form

$$S = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \qquad B(x) = \begin{pmatrix} B_1(x) & 0 \\ 0 & B_2(x) \end{pmatrix},$$

$$M(x) = \begin{pmatrix} 0 & A(x) \\ -A^T(x) & 0 \end{pmatrix},$$

the system (20) becomes the system (19). It means that the system (20) can be transformed to the scalar system (19). Certainly the square form

$$V = \langle Sy, y \rangle$$

derivative along the solutions of the system (20) has the following form

$$\dot{V} = \langle 2B(x)y, y \rangle . \tag{21}$$

Subsequently let us generalize the system (20) and replace the constant matrix S with a nondegenerated continuous matrix $S(x) \in C'(\mathbb{R}^m; f)$ in such a way the square form $V = \langle S(x)y, y \rangle$ derivative along the solutions of the system (20) will fulfills (21). When in the system (20) the matrix S will be replaced with $S(x)$ then (21) does not hold. Therefore let us consider the system (20) in the following form

$$\frac{dx}{dt} = f(x), \quad S(x)\frac{dy}{dt} = \left[B(x) + M(x) - \frac{1}{2}\dot{S}(x) \right] y, \quad y \in \mathbb{R}^k. \tag{22}$$

The square form $V = \langle S(x)y, y \rangle$ derivative along the solutions of the system (22) will fulfills (21).

On the other hand we can research the system (22) in a different way. Let a nondegenerated symmetric matrix $S(x) \in C'(\mathbb{R}^m; f)$ exists and (5) holds. Certainly the system (1) is equivalent to the system

$$\frac{dx}{dt} = f(x), \quad S(x)\frac{dy}{dt} = S(x)A(x)y,$$

and it is equivalent to

$$\frac{dx}{dt} = f(x), \quad S(x)\frac{dy}{dt} + \frac{1}{2}\dot{S}(x)y = \left[S(x)A(x) + \frac{1}{2}\dot{S}(x) \right] y. \tag{23}$$

In the system let us denote

$$\bar{A}(x) = S(x)A(x) + \frac{1}{2}\dot{S}(x).$$

Let us symetrize the matrix

$$\begin{aligned}
\bar{A}(x) = \frac{1}{2}\left(\bar{A}(x) + \bar{A}^T(x)\right) + \frac{1}{2}\left(\bar{A}(x) - \bar{A}^T(x)\right) = \\
= \frac{1}{2}\left\{\left[S(x)A(x) + \frac{1}{2}\dot{S}(x)\right] + \left[S(x)A(x) + \frac{1}{2}\dot{S}(x)\right]^T\right\} + \\
+ \frac{1}{2}\left\{\left[S(x)A(x) + \frac{1}{2}\dot{S}(x)\right] - \left[S(x)A(x) + \frac{1}{2}\dot{S}(x)\right]^T\right\} = \\
= \frac{1}{2}\left\{S(x)A(x) + A^T(x)S(x) + \dot{S}(x)\right\} + \frac{1}{2}\left\{S(x)A(x) - A^T(x)S(x)\right\}.
\end{aligned}$$

Let us denote

$$B(x) = \frac{1}{2}\left\{S(x)A(x) + A^T(x)S(x) + \dot{S}(x)\right\}, \tag{24}$$

$$M(x) = \frac{1}{2}\left\{S(x)A(x) - A^T(x)S(x)\right\}, \tag{25}$$

and the system (23) becomes as follows

$$\frac{dx}{dt} = f(x), \quad S(x)\frac{dy}{dt} + \frac{1}{2}\dot{S}(x)y = \left[B(x) + M(x)\right]y,$$

where the matrix $B(x)$ is in the form (24) and is positive definite. The skew-symmetric matrix $M(x)$ is in the form (25). When we transpose $\frac{1}{2}\dot{S}(x)y$ to right side we get the system (22).

Let us research the system (22) as individual system with a symmetric matrix $S(x) \in C'(\mathbb{R}^m; f)$ which fulfills the assumptions:

$$\det S(x) \neq 0 \quad \forall x \in \mathbb{R}^m, \quad \|S^{-1}(x)\| \leqslant \text{const} < \infty.$$

Let us consider the system (22) regularity with this assumptions.

From the equality (21) for the square form $V = \langle S(x)y, y\rangle$, $y \in \mathbb{R}^k$ the remark is true.

Remark 6. When in the system (22) the symmetric matrix $B(x) \in C^0(\mathbb{R}^m)$ is positive definite $\langle B(x)y, y\rangle \geqslant \beta\|y\|^2$, $\beta = \text{const} > 0$ or negative definite $\langle B(x)y, y\rangle \leqslant -\beta\|y\|^2$, $\beta = \text{const} > 0$ the system (22) is regular for every skew–symmetric matrix $M(x) \in C^0(\mathbb{R}^m)$.

An auxiliary lemma holds true.

Lemma 7. *Let $k \times k$ – dimensional symmetric matrix $B(x) \in C^0(\mathbb{R}^m)$ fulfills the following inequality*

$$\langle B(x)y, y\rangle \geqslant 0 \quad \forall y \in \mathbb{R}^k, \quad x \in \mathbb{R}^m, \tag{26}$$

then for every fixed $k \times k$–dimensional matrix $\Psi(x) \in C^0(\mathbb{R}^m)$ parameter $p > 0$ is sufficiently big that the square form

$$V = \|y\|^2 + p\,\langle B(x)y, y\rangle + \langle B(x)y, \Psi(x)y\rangle\,, \qquad y \in \mathbb{R}^k, \tag{27}$$

is positive definite.

Proof. Let us consider the square form

$$\Phi_p = \|y\|^2 + p\,\langle B(x)y, y\rangle + 2\,\langle B(x)y, \Psi(x)z\rangle + \|z\|^2, \qquad x, y \in \mathbb{R}^k. \tag{28}$$

Let us show that the square form is positive definite when the parameter $p > 0$ is sufficiently big. The square form (28) can be rearranged

$$\begin{aligned}\Phi_p = \langle (I_n + pB(x))\,y, y\rangle + 2\left\langle \Psi^T(x)B(x)y, z\right\rangle + \|z\|^2 = \\ = \langle \Gamma_p\,(y + Kz)\,, (y + Kz)\rangle + \|z\|^2 - \langle \Gamma_p Kz, Kz\rangle\,,\end{aligned} \tag{29}$$

where

$$\Gamma_p = I_k + pB(x), \quad K = \Gamma_p^{-1}B(x)\Psi(x). \tag{30}$$

From the equality (29) we have

$$\Phi_p \geqslant \|y + Kz\|^2 + \left(1 - \|K^T\Gamma_p K\|_0\right)\|z\|^2.$$

Let us notice that the square form (28) is positive definite when the condition holds true

$$\|K^T\Gamma_p K\|_0 < 1$$

and the parameter $p > 0$ is sufficiently big. Let us remind sufficient notation

$$\|K^T\Gamma_p K\|_0 = \sup_{x\in\mathbb{R}^m} \|K^T(x)\Gamma_p(x)K(x)\|.$$

From (30) follows

$$K^T\Gamma_p K = \Psi^T(x)B(x)\Gamma_p^{-1}(x)B(x)\Psi(x). \tag{31}$$

For sufficiently big $p > 0$ the following inequality holds true

$$\|\Gamma_p^{-1} B\|_0 \leqslant \frac{1}{p}. \tag{32}$$

Let us fix $x = x_0 \in \mathbb{R}^m$ and transform the matrix $B(x_0) = B$ into diagonal form

$$Q^{-1} B Q = \operatorname{diag}\{\beta_1, \ldots, \beta_k\}, \tag{33}$$

where Q – orthogonal matrix, $Q^T = Q^{-1}$. From the inequality (26) follows that $\beta_j \geqslant 0$, $j = \overline{1,k}$. By (33) we have the product $\Gamma_p^{-1} B$:

$$\begin{aligned}\Gamma_p^{-1} B = \frac{1}{p}\left(I_k + pB\right)^{-1}(pB) = \\ = \frac{1}{p} Q \operatorname{diag}\left\{\frac{1}{1+p\beta_1}, \ldots, \frac{1}{1+p\beta_k}\right\} Q^{-1} \cdot Q \operatorname{diag}\left\{p\beta_1, \ldots, p\beta_k\right\} Q^{-1} = \\ = \frac{1}{p} Q \operatorname{diag}\left\{\frac{p\beta_1}{1+p\beta_1}, \ldots, \frac{p\beta_k}{1+p\beta_k}\right\} Q^{-1}.\end{aligned} \tag{34}$$

We have (by $\|Q\| = \|Q\|^{-1} = 1$ and (34)):

$$\|\Gamma_p^{-1} B\| \leqslant \frac{1}{p}\left\|\operatorname{diag}\left\{\frac{p\beta_1}{1+p\beta_1}, \ldots, \frac{p\beta_k}{1+p\beta_k}\right\}\right\| < \frac{1}{p}.$$

Let us transpose $x = x_0 \in \mathbb{R}^m$ in the inequality (32). From $\beta_j \geqslant 0$ (31) and inequality (32) we have

$$\|K^T \Gamma_p K\| \leqslant \|\Psi^T(x)\| \cdot \|B(x)\| \cdot \|\Gamma_p^{-1}(x) B(x)\| \cdot \|\Psi(x)\| \leqslant \frac{1}{p}\|\Psi\|_0^2 \|B\|_0.$$

When the parameter $p \geqslant 2\|\Psi\|_0^2 \|B\|_0$ then $\|K^T \Gamma_p K\|_0 \leqslant 0,5$. The square form (28) derivative is then positive definite

$$\Phi_p = \|y\|^2 + p\left\langle B(x)y, y\right\rangle + 2\left\langle B(x)y, \Psi(x)z\right\rangle + \|z\|^2 \geqslant \epsilon\left(\|x\|^2 + \|y\|^2\right),$$

$\epsilon = \text{const} > 0$. In the inequality let us replace $z = y$:

$$2\|y\|^2 + p\left\langle B(x)y, y\right\rangle + 2\left\langle B(x)y, \Psi(x)y\right\rangle \geqslant 2\epsilon\|y\|^2, \qquad \epsilon = \text{const} > 0.$$

Let us replace $p \to 2p$ and we get positive definite square form (27). $\square$

The following theorem holds true.

Theorem 8. *Let the system(22) with $B(x) \equiv 0$:*

$$\frac{dx}{dt} = f(x), \quad S(x)\frac{dy}{dt} = \left[M(x) - \frac{1}{2}\dot{S}(x)\right] y \tag{35}$$

possesses at least one Green function (4) then the Green function is unique, the number of variable y ($y \in \mathbb{R}^k$) is even $k = 2n$ and the system (22) is regular for every symmetric matrix $B(x) \in C^0(\mathbb{R}^m)$ which satisfies (26).

Proof. Let us denote

$$N(x) = S^{-1}(x)\left[M(x) - \frac{1}{2}\dot{S}(x)\right]. \tag{36}$$

Because the system (35) possesses at least one Green function the square form $\langle \Theta(x)z, z\rangle = V$, $\Theta(x) \equiv \Theta^T(x) \in C'(\mathbb{R}^m; f)$ exists and its derivative along the conjoint system to (35):

$$\frac{dx}{dt} = f(x), \quad \frac{dz}{dt} = -N^T(x)z$$

is positive definite. That is the inequality holds true

$$\dot{V} = \left\langle \left[\dot{\Theta}(x) - \Theta(x)N^T(x) - N(x)\Theta(x)\right] z, z\right\rangle \geqslant \|z\|^2. \tag{37}$$

We note that the square form

$$\langle S(x)\Theta(x)S(x)y, y\rangle = W$$

derivative along the solutions of the system (35) is also positive definite. Let us change the variables $z = S(x)y$ in the equality (37). We have

$$\begin{aligned}
S\left\{\dot{\Theta} - \Theta N^T - N\Theta\right\} &= S\left\{\dot{\Theta} + \Theta\left[M + \frac{1}{2}\dot{S}\right]S^{-1} + S^{-1}\left[-M + \frac{1}{2}\dot{S}\right]\Theta\right\}S = \\
&= S\dot{\Theta}S + \frac{1}{2}S\Theta\dot{S} + \frac{1}{2}\dot{S}\Theta S + S\Theta M - M\Theta S = \\
&= S\dot{\Theta}S + \dot{S}\Theta S + S\Theta\dot{S} - \frac{1}{2}S\Theta\dot{S} - \frac{1}{2}\dot{S}\Theta S + S\Theta M - M\Theta S = \\
&= \dot{S}\Theta S + S\dot{\Theta}S + S\Theta\dot{S} + S\Theta S\left\{S^{-1}\left[M - \frac{1}{2}\dot{S}\right]\right\} + \left\{S^{-1}\left[M - \frac{1}{2}\dot{S}\right]\right\}^T S\Theta S.
\end{aligned}$$

That is the square form derivative along the solutions of the system (35) is positive definite

$$\dot{W} \geqslant \|S(x)y\|^2 \geqslant \frac{1}{\|S^{-1}\|_0^2}\|y\|^2 = \epsilon\|y\|^2.$$

It means the two square forms V and W existence follows $\det\Theta \neq 0$ $\forall x \in \mathbb{R}^m$ and the system (35) is regular.

Let us proof that in regular system (35) the number of variables k is even. When the system (35) is regular then the linear system

$$\frac{dy}{dt} = N^T(x(t;x))y \tag{38}$$

is exponentially dichotomic on the axis $\mathbb{R}$. Let us assume the system (38) has r linear independent solutions which approach zero $+\infty$ and $k-r$ linear independent solutions which approach zero in $-\infty$. Then the conjoint system

$$\frac{dz}{dt} = -N^T(x(t;x))z \tag{39}$$

has r linear independent solutions which approach zero in $-\infty$ and $k-r$ linear independent solutions which approach zero in $+\infty$. On the other hand from (36) and skew-symmetric matrix $M(x)$ the systems (38) and (39) satisfy the identity $S(x(t;x)) \cdot y(t) \equiv z(t)$. It means that both systems (38) and (39) have identical number of solution which approach to zero in $+\infty$. Therefore k is even and $k-r=r$.

Let us proof the system (22) regularity for symmetric matrix $B(x) \in C^0(\mathbb{R}^m)$ which fulfills (26). Let us consider the square form with positive parameter p:

$$V_p = p\left\langle S(x)y, y\right\rangle + \left\langle S(x)\Theta(x)S(x)y, y\right\rangle$$

and let us show that its derivative along the solutions of the system (22) is positive definite for sufficiently big $p > 0$. Let denote $\Theta_1(x) = S(x)\Theta(x)S(x)$ and by (36) we have

$$\begin{aligned}
\dot{V}_p = 2p\left\langle B(x)y, y\right\rangle + \Big\langle \Big[\dot{\Theta}_1(x) + \Theta_1(x)\left(S^{-1}(x)B(x) + N(x)\right) + \\
+ \left(B(x)S^{-1}(x) + N^T(x)\right)\Theta_1(x)\Big]y, y\Big\rangle = \\
= \Big\langle \Big(\dot{\Theta}_1(x) + \Theta_1 N + N^T\Theta_1\Big)y, y\Big\rangle + 2p\left\langle B(x)y, y\right\rangle + \left\langle \Theta_1 S^{-1}By, y\right\rangle + \\
+ \left\langle BS^{-1}\Theta_1 y, y\right\rangle \geqslant \epsilon\|y\|^2 + 2p\left\langle By, y\right\rangle + 2\left\langle By, S^{-1}\Theta_1 y\right\rangle.
\end{aligned}$$

By the lemma the derivative $\dot{V}_p$ is positive definite for sufficiently big $p > 0$. That is the system (22) is regular. □

Remark 9. The theorem holds true when matrix $B(x)$ is negative definite $\langle B(x)y, y\rangle \leqslant 0$, $\forall y \in \mathbb{R}^k, x \in \mathbb{R}^m$.

Remark 10. In case in the system (22) the matrix $B(x)$ is not symmetric, the condition (26) is not sufficient to regularity of system (22) when the system (35) is regular.

Let us consider the system which fulfills the theorem assumptions

$$\frac{dx}{dt} = f(x),$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \tag{40}$$

$$= \left\{ \begin{bmatrix} \beta_1 \cos^2 x & 0 \\ 0 & \beta_2 \cos^4 x \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

where $f(x) \in C_{Lip}(\mathbb{R}), \beta_1 > 0, \beta_2 > 0$. All theorem assumptions hold so the system (40) is regular for every function $f(x) \in C_{Lip}(\mathbb{R})$. On the other hand we can proof the system (40) regularity by the Lyapunov function:

$$V_p = p y_1 y_2 + y_1^2 - y_2^2.$$

Remark 11. The theorem contains sufficient conditions for the system (22) regularity. We can show that the theorem does not hold true and the system (22) is regular.

Let us consider the example

$$\frac{dx}{dt} = 1,$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \tag{41}$$

$$= \left\{ \begin{bmatrix} \beta_1 \cos^2 x & 0 \\ 0 & \beta_2 \cos^4 x \end{bmatrix} + \begin{bmatrix} 0 & -\sin x \\ \sin x & 0 \end{bmatrix} \right\} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

where $\beta_1 > 0$, $\beta_2 > 0$. In the example the system corresponding to (35) is in the following form

$$\frac{dx}{dt} = 1,$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & -\sin x \\ \sin x & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

and it is not regular because the heterogeneous system

$$\frac{dy_1}{dt} = y_1 \sin t + h_1(t), \quad \frac{dy_2}{dt} = -y_2 \sin t + y_2(t),$$

has bounded on $\mathbb{R}$ solution for every function $(h_1(t), h_2(t))$. Besides when the conditions hold $\beta_1 > 0$, $\beta_2 > 0$ the system (41) is regular. To check the system regularity let us consider the square form:

$$V = py_1y_2 + \left(y_1^2 - y_1^2\right)\sin x,$$

where $p > 0$.

Remark 12. Let us notice that the system (41) has the following form

$$\frac{dx}{dt} = 1$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} =$$

$$= \left\{ \begin{bmatrix} \beta_1 & 0 \\ 0 & 0 \end{bmatrix} \cos^2 x + \begin{bmatrix} 0 & 0 \\ 0 & \beta_2 \end{bmatrix} \cos^4 x + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \sin x \right\} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Because the theorem assumptions do not hold for the system it is interesting to research the following system

$$\frac{dx}{dt} = f(x), \quad S\frac{dy}{dt} = \left[\sum_{j=1}^{k} B_j(x) \cdot \nu_j(x) + M(x)\mu(x)\right] y, \tag{42}$$

with symmetric continuos matrix S, $f(x) \in C_{Lip}(\mathbb{R}^m)$.

Let the symmetric matrix $B(x) \in C^0(\mathbb{R}^m)$ in the system (42) fulfills

$$\begin{aligned} &\langle B_j(x)y, y\rangle \geqslant 0, \qquad j = \overline{1,k}, \qquad \forall x \in \mathbb{R}^m, \quad y \in \mathbb{R}^n, \\ &\sum_{j=1}^{k} \langle B_j(x)y, y\rangle \geqslant \beta\|y\|^2, \quad \beta = \text{const} > 0. \end{aligned} \tag{43}$$

The matrix $M(x) \in C^0(\mathbb{R}^m)$ is skew-symmetric. Scalar functions $\nu_j(x) \in C^0(\mathbb{R}^m)$, $j = \overline{1,k}$ and the condition holds

$$\nu_j(x) \geqslant 0, \quad j = \overline{1,k} \quad \forall x \in \mathbb{R}^m.$$

For example $\cos^2 x$, $|\cos x|$, $\cos^4 x$, $\frac{1}{\operatorname{ch} x}$, $\left(\frac{1}{\operatorname{ch} x}\right)^n$, $\sin^{2n} x_1 \cos^{2k} x_2$ itd. The scalar function $\mu(x)$ is bounded in $\mathbb{R}^m$, it has continous derivative and all first partial derivatives are continuous and bounded in $\mathbb{R}^m$, that is $\mu(x) \in C^1(\mathbb{R}^m) \cap C^0(\mathbb{R}^m)$, $\dfrac{\partial \mu(x)}{\partial x_i} \in C^0(\mathbb{R}^m)$. For example $\sin x$, $(\sin x)^{2k-1}$, $\operatorname{th} x$, $(\operatorname{th} x)^{2k-1}$ etc.

Let us select the scalar function $\nu_j(x)$, $\mu(x)$ to the inequality holds true

$$p\nu_0(x) + \mu^2(x) - K \sum_{i=1}^{m} \left| \frac{\partial \mu(x)}{\partial x_i} \right| - L\bar{\nu}(x) \geqslant \epsilon, \quad \epsilon = \mathrm{const} > 0, \tag{44}$$

for every constant K, L and sufficiently big parameter $p > 0$ and $x \in \mathbb{R}^m$ where $\nu_0(x) = \min\{\nu_1(x), \ldots, \nu_k\}$, $\bar{\nu}(x) = \max\{\nu_1(x), \ldots, \nu_k(x)\}$. Let $\nu_1(x) = \cos^8 x$, $\nu_2(x) = \cos^{10} x$, $\mu(x) = \sin^3 x$ and the inequality (44) holds true for sufficiently big parameter $p > 0$.

The following theorem holds true.

Theorem 13. *Let a constant symmetric $n \times n$ – dimensional matrix Θ exists and the inequality holds true*

$$\left\langle \left[\Theta S^{-1} M(x) - M(x) S^{-1} \Theta(x)\right] y, y \right\rangle \geqslant \|y\|^2, \quad \forall y \in \mathbb{R}^n.$$

When the conditions (43) – (44) hold the system (42) is regular for every fixed bounded function $f(x) \in C_{Lip}(\mathbb{R}^m) \cap C^0(\mathbb{R}^m)$. Moreover the square form

$$V = p\left\langle Sy, y\right\rangle + \left\langle \Theta y, y\right\rangle \cdot \mu(x) \tag{45}$$

derivative along the solutions of the system (42) is positive definite for sufficiently big parameter $p > 0$.

Proof. The square form (45) derivative along the solutions of the system (42):

$$\begin{aligned}
\dot{V} = 2p \sum_{j=1}^{k} \left\langle B_j(x) y, y\right\rangle \cdot \nu_j(x) + \\
+ 2\left\langle \Theta y, S^{-1} \left[\sum_{j=1}^{k} B_j(x) \cdot \nu_j(x) + M(x)\mu(x)\right] y \right\rangle \mu(x) + \left\langle \Theta y, y\right\rangle \dot{\mu}(x) \geqslant \\
\geqslant 2p\beta \|y\|^2 \nu_0(x) + 2\left\langle \Theta y, S^{-1} \left[\sum_{j=1}^{k} B_j(x) \nu_j(x)\right] y \right\rangle \mu(x) + \\
+ 2\left\langle \Theta y, S^{-1} M(x) y\right\rangle \mu^2(x) + \left\langle \Theta y, y\right\rangle \dot{\mu}(x),
\end{aligned} \tag{46}$$

where $\dot{\mu}(x) = \sum_{i=1}^{m} \frac{\partial \mu(x)}{\partial x_i} f_i(x)$. Let us estimate every summand in the inequality (46):

$$2\left\langle \Theta y, S^{-1}B_j(x)y\right\rangle \nu_j(x)\mu(x) \geqslant \\ \geqslant -2\left|\left|\Theta S^{-1}B_j\right|\right|_0 \|y\|^2\nu_j(x)\|\mu\|_0 \geqslant -L_0\nu_j(x)\|y\|^2,$$

$$2\left\langle \Theta y, S^{-1}M(x)y\right\rangle \mu^2(x) \geqslant \|y\|^2\mu^2(x),$$

$$\langle \Theta y, y\rangle \dot{\mu}(x) \geqslant -\|\Theta\| \cdot \dot{\mu}(x) \cdot \|y\|^2 \geqslant -K_1 \sum_{i=1}^{m} \left|\frac{\partial \mu}{\partial x_i}\right| \|y\|^2,$$

where L_0, K_1 – positive constant. We have

$$\dot{V} \geqslant \left(2p\beta\nu_0(x) + \mu^2(x) - K_1 \sum_{i=1}^{m} \left|\frac{\partial \mu}{\partial x_i}\right| - L\bar{\nu}(x)\right) \|y\|^2.$$

By (44) we can select the scalar function $\nu_j(x)$, $\mu(x)$ that for every constant K, L and sufficiently big parameter $p > 0$ the derivative $\dot{V}$ is positive definite. Then the system (42) is regular. □

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Omówienie

W artykule przeprowadzono badania regularności liniowych rozszerzeń układów dynamicznych. Wykorzystano metodę funkcji Lapunowa o zmiennym znaku, która jest niezwykle efektywnym narzędziem. Ponadto, przedstawiono metodę doprowadzenia układów słabo regularnych do regularnych. Przestawiono warunki regularności układu o postaci:

$$\frac{dx}{dt} = f(x), \qquad S(x)\frac{dy}{dt} = \left[M(x) - \frac{1}{2}\dot{S}(x)\right] y.$$

Podano konstrukcję formy kwadratowej:

$$V = p\,\langle Sy, y\rangle + \langle \Theta y, y\rangle \cdot \mu(x)$$

za pomocą, której udowodniono regularność układu:

$$\frac{dx}{dt} = f(x), \qquad S\,\frac{dy}{dt} = \left[\sum_{j=1}^{k} B_j(x)\cdot \nu_j(x) + M(x)\mu(x)\right] y.$$