ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 2011

Seria: MATEMATYKA STOSOWANA z. 1 Nr kol. 1854

Dariusz PĄCZKO

Institute of Mathematics
Silesian University of Technology

SOME CONSTRUCTIONS OF LINEAR EXTENSIONS OF DYNAMICAL SYSTEMS

Summary. The paper presents a method to transform two weakly regular systems to one regular system. Additionally, it is proved that there is no linear change of variables reducing the extension of a dynamical system to a conjugate system where the number of normal variables is three.

PEWNE KONSTRUKCJE LINIOWYCH ROZSZERZEŃ UKŁADÓW DYNAMICZNYCH

Streszczenie. W artykule przedstawiono metodę doprowadzenia dwóch układów słabo regularnych do jednego układu regularnego. Dodatkowo pokazano, że niemożliwe jest doprowadzenie liniowego rozszerzenia układu dynamicznego do układu sprzężonego, w przypadku gdy liczba zmiennych normalnych wynosi trzy.

2010 Mathematics Subject Classification: 34D10, 34D35, 37L25.
Wpłynęło do Redakcji (received): 10.07.2011 r.

1. Introduction

Consider the system of differential equations

$$\begin{cases} \dfrac{d\psi}{dt} = \omega(\psi), \\ \dfrac{dx}{dt} = A(\psi)x, \end{cases} \tag{1}$$

where $t \in \mathbb{R}$, $\psi \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $\omega(\psi) = (f_1(\psi), \dots, f_m(\psi))$ is a vector function defined for all $\psi \in \mathbb{R}^m$, satisfying locally the Lipschitz condition

$$\|\omega(\psi) - \omega(\overline{\psi})\| \leqslant L\|\psi - \overline{\psi}\|,$$

where the constant L depends on the area in which there are points of ψ, $\overline{\psi}$. In addition, we assume that the function $\omega(\psi)$ satisfies the inequality

$$\|\omega(\psi)\| \leqslant \alpha_1\|\psi\| + \alpha_2,$$

for all $\psi \in \mathbb{R}^m$ with some non-negative constants α_1, α_2. The space of such functions will be denoted by $C_{Lip}(\mathbb{R}^m)$.

Assumptions made allow to conclude that the Cauchy problem

$$\begin{cases} \dfrac{d\psi}{dt} = \omega(\psi), \\ \psi\big|_{t=0} = \psi_0, \end{cases}$$

has a unique solution $\psi = \psi(t; \psi_0)$ for each fixed $\psi_0 \in \mathbb{R}^m$, defined for all $t \in \mathbb{R}$.

$A(\psi)$ is square $n \times n$ dimensional matrix, elements of which are scalar real functions, defined, continuous and bounded on $\mathbb{R}^m$.

In what follows we will use the following notation:

- $C^0(\mathbb{R}^m)$ means the space of real functions, continuous and bounded on $\mathbb{R}^m$,
- $\langle x, \overline{x}\rangle = \sum\limits_{j=1}^{n} x_j\overline{x}_j$ means the scalar product in $\mathbb{R}^n$,
- $\|x\| = \sqrt{\langle x, x\rangle}$ means the norm of $x \in \mathbb{R}^n$,
- $\|A\| = \max\limits_{\|x\|=1}\|Ax\|$,
- $\|A\|_0 = \sup\limits_{\psi\in\mathbb{R}^m}\|A(\psi)\|$,

- $\Omega^t_\tau(\psi_0)$ is a fundamental matrix of a linear system of equations

$$\frac{dx}{dt} = A(\psi(t;\psi_0))\,x, \tag{2}$$

normed at the $t = \tau$, i.e., $\Omega^t_\tau(\psi_0)\big|_{t=\tau} = I_n$, where I_n is $n \times n$ dimensional unit matrix,

- $C'(\mathbb{R}^m, \omega)$ denotes the subspace of $C^0(\mathbb{R}^m)$ consisting of all functions $F(\psi)$, such that the function $F(\psi(t,\psi))$ is continuously differentiable function of $t \in \mathbb{R}$, where

$$\dot{F}(\psi) \overset{df}{=} \frac{d}{dt} F(\psi(t;\psi))\bigg|_{t=0}, \quad \dot{F}(\psi) \in C^0(\mathbb{R}^m).$$

In order to simplify the notation, subscript zero at ψ_0 in the solution of the Cauchy problem $\psi = \psi(t;\psi_0)$ will be omitted.

Together with the system (1) we consider heterogeneous system of equations

$$\begin{cases} \dfrac{d\psi}{dt} = \omega(\psi), \\ \dfrac{dx}{dt} = A(\psi)x + h(\psi), \end{cases} \tag{3}$$

where $h(\psi) \in C^0(\mathbb{R}^m)$.

Definition 1. *We say that system (3) has a bounded invariant manifold, defined by equality*

$$x = u(\psi), \tag{4}$$

if the function $u(\psi) \in C'(\mathbb{R}^m, \omega)$ and satisfies the identity

$$\dot{u}(\psi) \equiv A(\psi)u(\psi) + h(\psi) \qquad \forall \psi \in \mathbb{R}^m. \tag{5}$$

Definition 2. *Let $C(\psi) \in C^0(\mathbb{R}^m)$ be $n \times n$-dimensional matrix, such that the function $G_0(\tau,\psi)$ of the form*

$$G_0(\tau,\psi) = \begin{cases} \Omega^0_\tau(\psi)C(\psi(\tau;\psi)), & \tau \leqslant 0, \\ \Omega^0_\tau(\psi)\Big[C(\psi(\tau;\psi)) - I_n\Big], & \tau > 0, \end{cases} \tag{6}$$

satisfied the estimation

$$\|G_0(\tau,\psi)\| \leqslant K e^{-\gamma|\tau|}, \tag{7}$$

where K and γ are certain positive constants. Then the function defined by (6) is said to be the Green function of problem of bounded invariant manifold of the system (1).

The existence of the Green function (6) implies that the system (3) has a bounded invariant manifold (4) for each fixed function $h(\psi) \in C^0(\mathbb{R}^m)$, which can be written in the integral form

$$x = u(\psi) = \int_{-\infty}^{\infty} G_0(\tau, \psi)\, h(\psi(\tau; \psi))\, d\tau. \tag{8}$$

To see this, write down the composition

$$\begin{aligned} u(\psi(t;\psi)) &= \int_{-\infty}^{\infty} G_0(\tau, \psi(t;\psi))\, h(\psi(\tau, \psi(t;\psi)))\, d\tau = \\ &= \int_{-\infty}^{\infty} G_t(t+\tau, \psi)\, h(\psi(t+\tau, \psi))\, d\tau = \int_{-\infty}^{\infty} G_t(\tau, \psi)\, h(\psi(\tau;\psi))\, d\tau, \end{aligned}$$

where

$$G_t(\tau, \psi) = \begin{cases} \Omega_\tau^t(\psi) C(\psi(\tau;\psi)), & \tau \leqslant t, \\ \Omega_\tau^t(\psi)\Big[C(\psi(\tau;\psi)) - I_n\Big], & \tau > t. \end{cases} \tag{9}$$

We obtain

$$\begin{aligned} u(\psi(t;\psi)) = \int_{-\infty}^{t} & \Omega_\tau^t(\psi) C(\psi(\tau;\psi))\, h(\psi(\tau;\psi))\, d\tau + \\ & + \int_{t}^{+\infty} \Omega_\tau^t(\psi)\Big[C(\psi(\tau;\psi)) - I_n\Big]\, h(\psi(\tau;\psi))\, d\tau. \end{aligned}$$

In this way, the function $x = u(\psi(t, \psi))$ is a bounded solution for the heterogeneous system of equations

$$\frac{dx}{dt} = A(\psi(t;\psi))x + h(\psi(t;\psi)),$$

which implies the identity (5).

Remark 3. Because the structure of the Green function (6) is not included the function $h(\psi) \in C^0(\mathbb{R}^m)$, the existence of such a function depends only on the system (1).

Remark 4. The estimation (7) is equivalent to the inequality

$$\|G_t(0, \psi)\| \leqslant K e^{-\gamma|t|} \tag{10}$$

for auxiliary function

$$G_t(0,\psi) = \begin{cases} \Omega_0^t(\psi)C(\psi), & t \geqslant 0, \\ \Omega_0^t(\psi)\Big[C(\psi) - I_n\Big], & t < 0. \end{cases} \tag{11}$$

Remark 5. If the system (1) has the Green function (6), the heterogeneous system

$$\frac{dx}{dt} = A(\psi(t;\psi))x + h(t), \tag{12}$$

has the solution bounded on the axis $\mathbb{R}$ for any fixed function $h(t)$ continuous (continuous on intervals) and bounded on $\mathbb{R}$. The solution $x = x(t,\psi)$ can be written in the integral form

$$x(t;\psi) = \int_{-\infty}^{+\infty} G_t(\tau,\psi)\, h(\tau)\, d\tau.$$

Remark 6. The function (9) is the Green function of problem of bounded solutions of $\frac{dx}{dt} = A(\psi(t,\psi))x$, for each fixed value of $\psi \in \mathbb{R}^m$.

Corollary 7. *If the homogeneous linear system (12) for some fixed value of $\psi \in \mathbb{R}^m$ and for certain function $h(t)$ continuous and bounded on $\mathbb{R}$ has not bounded solution, then the system (1) has no the Green function (6).*

Definition 8. *The system of equations (1) is called regular, if there exists a Green function (6). Only that there exists for at least one Green function, then the system (1) is called weakly regular. If there are infinitly many Green functions for the system (1), then the system is called strictly weakly regular.*

Alike issues were already brought up in researches [1–5].

2. The constructions of regular linear extensions of dynamical system

Consider two systems of differential equations

$$\begin{cases} \dfrac{d\psi}{dt} = \omega_1(\psi), \\ \dfrac{dx}{dt} = A_1(\psi)x, \end{cases} \tag{13}$$

$$\begin{cases} \dfrac{d\psi}{dt} = \omega_2(\psi), \\ \dfrac{dx}{dt} = A_2(\psi)x, \end{cases} \tag{14}$$

where $x \in \mathbb{R}^n$, $\psi \in \mathbb{R}^m$, $\omega_i(\psi) \in C_{Lip}(\mathbb{R}^m)$, $A_i(\psi) \in C^0(\mathbb{R}^m)$.

Theorem 9. *If the systems (13) and (14) are weakly regular, then the system*

$$\begin{cases} \dfrac{d\psi_1}{dt} = \omega_1(\psi_1), \\ \dfrac{d\psi_2}{dt} = \omega_2(\psi_2), \\ \dfrac{dx_1}{dt} = [A_2(\psi_2) + \frac{1}{2}(A_1(\psi_1) + A_1^T(\psi_1)) - I_n]x_1 + \\ \qquad\qquad + [A_2^T(\psi_1) + A_1(\psi_1)]x_2, \\ \dfrac{dx_2}{dt} = [-A_2(\psi_2) + \frac{1}{2}(A_1(\psi_1) - A_1^T(\psi_1)) + I_n]x_1 - A_2^T(\psi_2)x_2, \\ \dfrac{dx_3}{dt} = [A_2(\psi_2) + \frac{1}{2}(A_1^T(\psi_1) - A_1(\psi_1)) + I_n]x_1 - \\ \qquad\qquad - [A_1(\psi_1) + A_2^T(\psi_2)]x_2 - A_1^T(\psi_1)x_3 \end{cases} \tag{15}$$

where $x_i \in \mathbb{R}^n$, $\psi_i \in \mathbb{R}^m$, $\omega_i(\psi_i) \in C_{Lip}(\mathbb{R}^m)$, $A_i(\psi_j) \in C^0(\mathbb{R}^m)$, is regular, i.e., has exactly one $3n \times 3n$ dimensional Green function $G_0(\tau, \psi_i, \psi_2)$.

The derivative of a quadratic form

$$V_p = p^2\{\langle x_1,\, x_2\rangle + \langle x_1,\, x_3\rangle + \langle x_2,\, x_3\rangle\} + \\ + p\langle S_2(\psi_2)x_2,\, x_2\rangle + \langle S_1(\psi_1)x_3,\, x_3\rangle \tag{16}$$

with respect to system (15) for sufficiently large values of $p \gg 1$ is positive definite.

Proof. Because of the weak regularity of systems (13) and (14) there exist symmetric matrices $S_i(\psi) \in C'(\mathbb{R}^m, \omega)$, $i = 1, 2$, satisfying the inequality

$$\Big\langle [\dot{S}_i(\psi) - S_i(\psi)A_i^T(\psi) - A_i(\psi)S_i(\psi)]x,\, x\Big\rangle \geqslant \|x\|^2, \tag{17}$$

where $S_i(\psi)$ may be degenerate matrix (see [6]).

Let

$$V_p = p^2\{\langle x_1,\, x_2\rangle + \langle x_1,\, x_3\rangle + \langle x_2,\, x_3\rangle\} + p\langle S_2(\psi_2)x_2,\, x_2\rangle + \langle S_1(\psi_1)x_3,\, x_3\rangle$$

be a quadratic form with a parameter $p > 0$.

We will prove that the derivative of this form with respect to the solutions of the system (15) for sufficiently large values of the parameter $p > 0$, is positive definite.

Let us denote

$$v = \langle x_1,\, x_2\rangle + \langle x_1,\, x_3\rangle + \langle x_2,\, x_3\rangle. \tag{18}$$

Calculating the derivative of the form v with respect to system (15) we get

$$\dot v = 2\langle Ix_1,\, x_1\rangle.$$

Assuming

$$w = p\langle S_2(\psi_2)x_2,\, x_2\rangle + \langle S_1(\psi_1)x_3,\, x_3\rangle, \tag{19}$$

the derivative of this form with respect to solutions of the system (15) is equal to

$$\dot w = p\Big\{\langle \dot S_2 x_2,\, x_2\rangle - \langle S_2A_2^T x_2,\, x_2\rangle - \langle A_2S_2x_2,\, x_2\rangle\Big\} + \\ + 2p\Big\langle S_2[-A_2 + \frac{1}{2}(A_1 - A_1^T) + I]x_1,\, x_2\Big\rangle + \langle \dot S_1x_3,\, x_3\rangle - \langle S_1A_1^Tx_3,\, x_3\rangle - \\ - \langle A_1S_1x_3,\, x_2\rangle + 2\Big\langle S_1[A_2 + \frac{1}{2}(A_1^T - A_1) + I]x_1,\, x_3\Big\rangle - 2\langle S_1[A_1 + A_2^T]x_2,\, x_3\rangle.$$

Let

$$K_1 = \|S_2[-A_2 + \frac{1}{2}(A_1 - A_1^T) + I]\|_0,$$
$$K_2 = \|S_1[A_2 + \frac{1}{2}(A_1^T - A_1) + I]\|_0,$$
$$K_3 = \|S_1[A_1 + A_2^T]\|_0.$$

Using the inequality (17), we have

$$\dot w \geqslant p\|x_2\|^2 + \|x_3\|^2 - 2pK_1\|x_1\|\|x_2\| - 2K_2\|x_1\|\|x_3\| - 2K_3\|x_2\|\|x_3\|.$$

Since $\dot V_p = p^2\dot v + \dot w$, then the formula at estimate is true

$$\dot V_p \geqslant 2p^2\|x_1\|^2 + p\|x_2\|^2 + \|x_3\|^2 - 2pK_1\|x_1\|\|x_2\| - \\ - 2K_2\|x_1\|\|x_3\| - 2K_3\|x_2\|\|x_3\|. \tag{20}$$

Consider the right handside of inequality (20) as a quadratic form Φ in three variables t_1, t_2, t_3:

$$\Phi(t_1,t_2,t_3) = 2p^2t_1^2 + pt_2^2 + t_3^2 - 2pK_1t_1t_2 - 2K_2t_1t_3 - 2K_3t_2t_3,$$

the matrix of this form is as follows

$$T = \begin{bmatrix} 2p^2 & -pK_1 & -K_2 \\ -pK_1 & p & -K_3 \\ -K_2 & -K_3 & 1 \end{bmatrix}.$$

It is obvious that for sufficiently large values of the parameter $p > 0$ matrix T is positive definite, and thus the derivative of a quadratic form V_p with respect to solutions of the system (15) is positive definite for sufficiently large values of the parameter $p > 0$.

Now we will prove that the quadratic form (16) is positive definite for $p \gg 0$.

Write down the matrix of the quadratic form (16), we have

$$S_p = \begin{bmatrix} 0 & \frac{1}{2}p^2 I_n & \frac{1}{2}p^2 I_n \\ \frac{1}{2}p^2 I_n & pS_2(\psi_2) & \frac{1}{2}p^2 I_n \\ \frac{1}{2}p^2 I_n & \frac{1}{2}p^2 I_n & S_1(\psi_1) \end{bmatrix}. \tag{21}$$

S_p matrix can be written in the following form

$$S_p = p^2 J + p\bar{S}_2(\psi_2) + \bar{S}_1(\psi_1),$$

where

$$J = \frac{1}{2}\begin{bmatrix} 0 & I_n & I_n \\ I_n & 0 & I_n \\ I_n & I_n & 0 \end{bmatrix},$$

$$\bar{S}_1(\psi_1) = \operatorname{diag}(0, 0, S_1(\psi_1)),$$
$$\bar{S}_2(\psi_2) = \operatorname{diag}(0, S_2(\psi_2), 0).$$

We will prove that the matrix S_p^2 for sufficiently large values of the parameter p is positive definite.

Since $S_p^2 = p^4 J^2 + p^3(J\bar{S}_2(\psi_2) + \bar{S}_2(\psi_2)J) + p^2(J\bar{S}_1(\psi_1) + \bar{S}_2^2(\psi_2) + \bar{S}_1(\psi_1)J) + \bar{S}_1^2(\psi_1)$, then assuming $u = [u_1, u_2, u_3]$, $u_i \in \mathbb{R}^n$, we obtain

$$\begin{aligned} \left\langle S_p^2 u,\, u \right\rangle = p^4 \left\langle J^2 u,\, u \right\rangle + p^3 \left\langle [J\bar{S}_2(\psi_2) + \bar{S}_2(\psi_2)J]u,\, u \right\rangle + \\ + p^2 \left\langle [J\bar{S}_1(\psi_1) + \bar{S}_2^2(\psi_2) + \bar{S}_1(\psi_1)J]u,\, u \right\rangle + \left\langle \bar{S}_1^2(\psi_1)u,\, u \right\rangle. \end{aligned}$$

Let us estimate each component of $\left\langle S_p^2 u,\, u \right\rangle$, so

$$\left\langle J^2 u,\, u \right\rangle \geqslant \frac{1}{4}\left(\|x_1 + x_2 + x_3\|^2 + \|x_1\|^2 + \|x_2\|^2 + \|x_3\|^2\right) \geqslant \frac{1}{4}\|u\|^2,$$
$$\left\langle [J\bar{S}_2(\psi_2) + \bar{S}_2(\psi_2)J]u,\, u \right\rangle \geqslant -M_2\|u\|^2,$$
$$\left\langle [J\bar{S}_1(\psi_1) + \bar{S}_2^2(\psi_2) + \bar{S}_1(\psi_1)J]u,\, u \right\rangle \geqslant -M_1\|u\|^2,$$
$$\left\langle \bar{S}_1^2(\psi_1)u,\, u \right\rangle \geqslant -M_0\|u\|^2,$$

where $M_i = \text{const} > 0$ for $i = 0, 1, 2$. We, therefore, get estimate

$$\left\langle S_p^2 u,\, u\right\rangle \geqslant (\frac{1}{4}p^4 - p^3 M_2 - p^2 M_1 - M_0)\|u\|^2.$$

We can stand up that for sufficiently large values of the parameter $p > 0$ matrix S_p^2 is positive definite, and hence $\det S_p^2 \neq 0$, so $\det S_p \neq 0$ for all $\psi_1, \psi_2 \in \mathbb{R}^m$.

We have proved that the quadratic form (16) has positive definite derivative with respect to system (15) and matrix S_p of this form is non-degenerate for sufficiently large values of the parameter $p > 0$, so the system (15) is regular, i.e., has exactly one Green function $G_0(\tau, \psi_1, \psi_2)$. □

Example. For two weakly regular systems

$$\begin{cases} \dfrac{d\psi}{dt} = \sin\psi, \quad \psi \in \mathbb{R}, \\ \dfrac{dx}{dt} = 3(\cos\psi)x, \quad x \in \mathbb{R}, \end{cases} \qquad \begin{cases} \dfrac{d\psi}{dt} = 1, \quad \psi \in \mathbb{R}, \\ \dfrac{dx}{dt} = -(\operatorname{tgh}\psi)x, \quad x \in \mathbb{R}, \end{cases}$$

we will show that the system

$$\begin{cases} \dfrac{d\psi_1}{dt} = \sin\psi_1, \\ \dfrac{d\psi_2}{dt} = 1, \\ \dfrac{dx_1}{dt} = [-1 + 3\cos\psi_1 - \operatorname{tgh}\psi_2]x_1 + [3\cos\psi_1 - \operatorname{tgh}\psi_2]x_2, \\ \dfrac{dx_2}{dt} = [1 + \operatorname{tgh}\psi_2]x_1 + [\operatorname{tgh}\psi_2]x_2, \\ \dfrac{dx_3}{dt} = [1 - \operatorname{tgh}\psi_2]x_1 - [3\cos\psi_1 - \operatorname{tgh}\psi_2]x_2 - [3\cos\psi_1]x_3, \end{cases} \tag{22}$$

is regular.

Consider the quadratic form

$$V_p = p^2(x_1x_2 + x_1x_3 + x_2x_3) + px_2^2 \operatorname{tgh}\psi_2 - x_3^2\cos\psi_1$$

and assume that

$$v_1 = x_1x_2 + x_1x_3 + x_2x_3.$$

Then the derivative of v_1 with respect to solutions of the system (22) is equal to

$$\dot{v}_1 = 2x_1^2.$$

Similarly, calculating the derivative

$$v_2 = x_2^2 \operatorname{tgh}\psi_2$$

with respect to the system (22), we obtain

$$\begin{aligned}\dot{v}_2 &= \frac{1}{\operatorname{ctgh}^2 \psi_2} x_2^2 + 2(\operatorname{tgh} \psi_2) x_2 \{[1 + \operatorname{tgh} \psi_2] x_1 + [\operatorname{tgh} \psi_2] x_2\} = \\ &= \left[\frac{1}{\operatorname{ctgh}^2 \psi_2} + 2(\operatorname{tgh} \psi_2)^2\right] x_2^2 + 2 \operatorname{tgh} \psi_2 (1 + \operatorname{tgh} \psi_2) x_1 x_2 \geqslant \\ &\geqslant x_2^2 - 4|x_1||x_2|\end{aligned}$$

And finally, calculating the derivative of the form

$$v_3 = (-\cos \psi_1) x_3^2$$

with respect to the system (22), we have

$$\begin{aligned}\dot{v}_3 &= x_3^2 \sin^2 \psi_1 - 2x_3 \{[1 - \operatorname{tgh} \psi_2] x_1 - [3\cos\psi_1 - \operatorname{tgh} \psi_2] x_2 - [3 \cos \psi_1] x_3\} = \\ &= x_3^2 (\sin^2 \psi_1 + 6 \cos^2 \psi_1) - 2[1 - \operatorname{tgh} \psi_2] x_1 x_3 + 2[3 \cos \psi_1 - \operatorname{tgh} \psi_2] x_2 x_3 \geqslant \\ &\geqslant x_3^2 - 4|x_1||x_3| - 8|x_2||x_3|.\end{aligned}$$

Finally, the derivative of the quadratic form V_p with respect to the system (22) satisfies the inequality

$$V_p \geqslant 2p^2 x_1^2 + p x_2^2 - 4p|x_1||x_2| + x_3^2 - 4|x_1||x_3| - 8|x_2||x_3|.$$

Consider the right handside of the above inequality as a quadratic form Φ in three variables t_1, t_2, t_3:

$$\Phi(t_1, t_2, t_3) = 2p^2 t_1^2 + p t_2^2 - 4p t_1 t_2 + t_3^2 - 4t_1 t_3 - 8t_2 t_3.$$

The matrix of this form is the following

$$T = \begin{bmatrix} 2p^2 & -2p & -2 \\ -2p & p & -4 \\ -2 & -4 & 1 \end{bmatrix}.$$

The matrix T is positive definite for $p > 20$, hence the system (22) is regular for $p > 20$.

In the case of two weakly regular systems

$$\begin{cases} \dfrac{d\psi}{dt} = \omega(\psi), \\ \dfrac{dx}{dt} = A_i(\psi) x, \qquad i = 1, 2, \end{cases}$$

where $x \in \mathbb{R}^n$, $\psi \in \mathbb{R}^m$, $\omega(\psi) \in C_{Lip}(\mathbb{R}^m)$, $A_i(\psi) \in C^0(\mathbb{R}^m)$ we define the matrix

$$P(\psi) = \begin{bmatrix} A_2 + \frac{1}{2}(A_1 + A_1^T) - I_n & A_2^T + A_1 & 0 \\ -A_2 + \frac{1}{2}(A_1 - A_1^T) + I_n & -A_2^T & 0 \\ A_2 + \frac{1}{2}(A_1^T - A_1) + I_n & -[A_1 + A_2^T] & -A_1^T \end{bmatrix},$$

such that the system

$$\begin{cases} \dfrac{d\psi}{dt} = \omega(\psi), \\ \dfrac{dx}{dt} = P(\psi)x, \qquad x \in \mathbb{R}^{3n}, \end{cases} \tag{23}$$

is regular.

By the linear change of variable

$$x = Lz,$$

where L is any non-degenerate matrix, we obtain the system of the form

$$\begin{cases} \dfrac{d\psi}{dt} = \omega(\psi), \\ \dfrac{dz}{dt} = L^{-1}P(\psi)Lz, \qquad z \in \mathbb{R}^{3n}. \end{cases} \tag{24}$$

The question is whether or not there is a matrix L which satisfies equality

$$L^{-1}P(\psi)L = -P^T(\psi),$$

i.e., whether or not it is possible to perform a change of variables in (23) such that the system obtained is conjugated with respect to variable x.

Negative response to that question gives the following theorem.

Theorem 10. *There is no non-degenerate matrix for the system (23) lead to a conjugated system with respect to variables x, i.e., the system*

$$\begin{cases} \dfrac{d\psi}{dt} = \omega(\psi), \\ \dfrac{dy}{dt} = -P^T(\psi)y, \qquad y \in \mathbb{R}^{3n}. \end{cases} \tag{25}$$

Proof. Consider the quadratic form

$$V = \langle x_1, x_2 \rangle + \langle x_1, x_3 \rangle + \langle x_2, x_3 \rangle.$$

By the linear change of variables

$$\begin{cases} x_1 = z_1 + z_2 - z_3, \\ x_2 = z_1 - z_2 - z_3, \\ x_3 = z_3, \end{cases}$$

we obtain

$$V = \|z_1\|^2 - \|z_2\|^2 - \|z_3\|^2.$$

Thus the number of positive eigenvalues of the matrix of the quadratic form V is not equal to the number of negative eigenvalues, so the matrix L must be degenerate. □

References

1. Bogolyubov N.N., Mitropolski Y.A.: *Asymptotic methods in the theory of non-linear oscilations*. Gordon and Breach, New York 1961.
2. Bogolyubov N.N., Mitropolskij Y.A., Samojlenko A.M.: *Methods of accelerated convergence in nonlinear mechanics*. Springer-Verlag, Berlin 1976.
3. Diliberto S.P.: *Perturbation theorems for periodic surfaces. I – Definitions and main theorems*. Rend. Circ. Mat. Palermo, II. Ser. **9** (1960), 265–299.
4. Diliberto S.P.: *Perturbation theorems for periodic surfaces II*. Rend. Circ. Mat. Palermo, II. Ser. **10** (1961), 111–161.
5. Hale J.K.: *Oscilations in nonlinear systems*. McGraw–Hill, New York 1963.
6. Kulyk V.L., Mitropolski Y.A., Samojlenko A.M.: *Dichotomies and stability in nonautonomous linear systems*. Taylor & Francis, London 2003.

Omówienie

Artykuł dotyczy zagadnienia istnienia ograniczonych inwariantnych rozmaitości dla układów dynamicznych. Przedstawiono w niej problem regularności liniowych rozszerzeń układów dynamicznych, a więc badanie istnienia funkcji Greena dla układów równań różniczkowych o postaci:

$$\begin{cases} \dfrac{d\psi}{dt} = \omega(\psi), \\ \dfrac{dx}{dt} = A(\psi)x, \end{cases}$$

gdzie $t \in \mathbb{R}$, $\psi \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $A(\psi)$ jest macierzą kwadratową, której elementami są funkcje rzeczywiste, ciągłe i ograniczone na $\mathbb{R}^m$. Dodatkowo, dla funkcji wektorowej $\omega(\psi) = (f_1(\psi), \ldots, f_m(\psi))$ zakładamy spełnienie warunków zapewniających, że zadanie Cauchy'ego:

$$\begin{cases} \dfrac{d\psi}{dt} = \omega(\psi), \\ \psi\big|_{t=0} = \psi_0 \end{cases}$$

ma jedyne rozwiązanie $\psi = \psi(t;\psi_0)$ dla wszystkich $\psi_0 \in \mathbb{R}^m$, określone dla $t \in \mathbb{R}$.

W artykule podano metodę doprowadzenia dwóch układów słabo regularnych, tj. mających przynajmniej jedną funkcję Greena, do jednego układu regularnego. Dodatkowo pokazano, że w przypadku, kiedy $x \in \mathbb{R}^{3n}$ niemożliwe jest doprowadzenie układu o postaci:

$$\begin{cases} \dfrac{d\psi}{dt} = \omega(\psi), \\ \dfrac{dx}{dt} = P(\psi)x, \qquad x \in \mathbb{R}^{3n} \end{cases}$$

do układu sprzężonego względem zmiennej x, tzn. układu:

$$\begin{cases} \dfrac{d\psi}{dt} = \omega(\psi), \\ \dfrac{dy}{dt} = -P^T(\psi)y, \qquad y \in \mathbb{R}^{3n}. \end{cases}$$