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ON CONDITIONS IMPLYING THAT CONTINUOUS LINEAR FUNCTIONALS DOES NOT EXIST – PART I

Summary. In this paper we discuss some problems concerning the existence of continuous linear functionals in the real $L^2([0,1])$ space.

O WARUNKACH IMPLIKUJĄCYCH NIEISTNIENIE FUNKCJONAŁÓW LINIOWYCH CIĄGŁYCH – CZĘŚĆ I

Streszczenie. Niniejszy artykuł poświęcony jest omówieniu pewnych zagadnień dotyczących istnienia funkcjonałów liniowych ciągłych w rzeczywistej przestrzeni $L^2([0,1])$.

1. Basic idea and solution

An impulse encouraging for preparing this paper gave us the following simple approximation problem. For which $\alpha \in \mathbb{R}$ there exists a linear continuous functional $F : L^2([0,1]) \to \mathbb{R}$ such that for almost all $n \in \mathbb{N}$ we have $F(x^n) = n^{-\alpha}$ (see [2]).

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It turns out that the negative answer to this question, about not existing of such functional if $\alpha < \frac{1}{2}$, is easy to receive on the basis of the result given below.

Theorem 1. *The following relations hold true:*

a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^\alpha} \in L^2([0,1]) \iff \alpha > \frac{1}{2};$

b) $\displaystyle\sum_{n=1}^{\infty} x^{n^\alpha} \in L^2([0,1]) \iff \alpha > 2.$

Proof. a) We have the relation ($s, t \in \mathbb{N}$, $s \leqslant t$):

$$\int_0^1 \Big(\sum_{n=s}^{t} \frac{x^n}{n^\alpha} \Big)^2 dx = \int_0^1 \Big(\sum_{n=s}^{t} \frac{x^{2n}}{n^{2\alpha}} + 2 \sum_{s\leqslant k<l\leqslant t} \frac{x^{k+l}}{(k\cdot l)^\alpha} \Big) dx = \\ = \sum_{n=s}^{t} \frac{1}{n^{2\alpha}(2n+1)} + 2 \sum_{s\leqslant k<l\leqslant t} \frac{1}{(k+l+1)(k\cdot l)^\alpha}.$$

Surely, the series $\sum\limits_{n\in\mathbb{N}} \frac{1}{n^{2\alpha}(2n+1)}$ is convergent iff $\alpha \in (0, +\infty)$.

Let $\alpha \in \left(\frac{1}{2}, +\infty\right)$. Then we get

$$\sum_{k=s}^{+\infty} \sum_{l=k+1}^{+\infty} \frac{1}{(k+l+1)(k\cdot l)^\alpha} \leqslant \sum_{k=s}^{+\infty} \frac{1}{k^\alpha} \sum_{l=k+1}^{+\infty} \frac{1}{l^{1+\alpha}} \leqslant \\ \leqslant \sum_{k=s}^{+\infty} \frac{1}{k^\alpha} \int_k^{+\infty} \frac{dx}{x^{1+\alpha}} = \frac{1}{\alpha} \sum_{k=s}^{+\infty} \frac{1}{k^{2\alpha}} \xrightarrow[s\to+\infty]{} 0$$

because of the convergence of the series $\sum\limits_{k\in\mathbb{N}} k^{-2\alpha}$.

On the other hand, we obtain

$$\sum_{k=s}^{+\infty} \sum_{l=k+1}^{+\infty} \frac{1}{\sqrt{kl}(k+l+1)} \geqslant \sum_{k=s}^{+\infty} \sum_{l=k+1}^{+\infty} \frac{1}{\sqrt{kl}2l} = \\ = \sum_{k=s}^{+\infty} \frac{1}{2\sqrt{k}} \sum_{l=k+1}^{+\infty} l^{-\frac{3}{2}} \geqslant \sum_{k=s}^{+\infty} \frac{1}{2\sqrt{k}} \int_{k+1}^{+\infty} x^{-\frac{3}{2}} dx = \\ = \sum_{k=s}^{+\infty} \frac{1}{2\sqrt{k}} \big[-2x^{-\frac{1}{2}} \big]_{k+1}^{+\infty} = \sum_{k=s}^{+\infty} \frac{1}{\sqrt{k(k+1)}} = +\infty.$$

Accordingly, the series $\sum\limits_{n=1}^{+\infty} \frac{x^n}{\sqrt{n}}$ is divergent in $L^2([0,1])$.

Certainly, for $\alpha \in [0, \frac{1}{2}]$, we have

$$\sum_{k=s}^{+\infty} \sum_{l=k+1}^{+\infty} \frac{1}{(kl)^\alpha \, (k+l+1)} \geqslant \sum_{k=s}^{+\infty} \sum_{l=k+1}^{+\infty} \frac{1}{\sqrt{kl}(k+l+1)} = +\infty,$$

which finally completes the proof of the first part of the theorem.

b) We have the following relations

$$\Big\| \sum_{n=1}^{\infty} x^{n^\alpha} \Big\|^2_{L^2([0,1])} = \int_0^1 \Big(\sum_{k=1}^{\infty} x^{n^\alpha} \Big)^2 dx =$$

$$= \int_0^1 \Big(\sum_{n=1}^{\infty} \Big(x^{2n^\alpha} + 2 \sum_{1 \leqslant k < n} x^{k^\alpha + n^\alpha} \Big) \Big) dx =$$

(by Monotone Convergence Theorem [3])

$$= \sum_{n=1}^{\infty} \Big(\frac{1}{2n^\alpha + 1} + 2 \sum_{1 \leqslant k < n} \frac{1}{k^\alpha + n^\alpha + 1} \Big)$$

from which, it is possible to instantly derive, if $\sum\limits_{n=1}^{\infty} n^{-\alpha} = +\infty$ then $\big(\sum\limits_{n=1}^{\infty} x^{n^\alpha} \big) \notin L^2([0,1])$, that is, if $\alpha \leqslant 1$, then $\big(\sum\limits_{n=1}^{\infty} x^{n^\alpha} \big) \notin L^2([0,1])$. Let $\alpha > 1$. We shall estimate from below the sum of the following series

$$\sum_{n=1}^{\infty} \sum_{1 \leqslant k < n} \frac{1}{k^\alpha + n^\alpha + 1} = \sum_{k=1}^{\infty} \sum_{n=k+1}^{\infty} \frac{1}{k^\alpha + n^\alpha + 1} =$$

$$= \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \sum_{n=s(k+1)}^{(s+1)(k+1)-1} \frac{1}{k^\alpha + n^\alpha + 1} \geqslant \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \frac{k+1}{k^\alpha + (s+1)^\alpha (k+1)^\alpha + 1} \geqslant$$

$$\geqslant \sum_{k=1}^{\infty} \frac{1}{2(k+1)^{\alpha-1}} \Big(\sum_{s=1}^{\infty} \frac{1}{(s+1)^\alpha} \Big),$$

which means that if $\alpha \in (1,2]$, then the series $\sum\limits_{n=1}^{\infty} \sum\limits_{1 \leqslant k < n} \frac{1}{k^\alpha + n^\alpha + 1}$ is divergent. If $\alpha > 2$, then we obtain

$$\sum_{n=1}^{\infty}\sum_{1\leqslant k<n}\frac{1}{k^\alpha+n^\alpha+1}=\sum_{k=1}^{\infty}\sum_{s=1}^{\infty}\sum_{n=s(k+1)}^{(s+1)(k+1)-1}\frac{1}{k^\alpha+n^\alpha+1}\leqslant$$
$$\leqslant\sum_{k=1}^{\infty}\sum_{s=1}^{\infty}\frac{k+1}{k^\alpha+s^\alpha(k+1)^\alpha+1}\leqslant\sum_{k=1}^{\infty}\frac{1}{(k+1)^{\alpha-1}}\sum_{s=1}^{\infty}\frac{1}{s^\alpha}<\infty.$$

□

Directly from the subitem $a)$ of the above Theorem it results that if $F(x^n)=n^{-\beta}$, for some $\beta\in\mathbb{R}$ and for all $n\in\mathbb{N}$, $n\geqslant N$, then also

$$F\Big(\frac{x^n}{n^\alpha}\Big)=n^{-\alpha-\beta},\quad n\geqslant N,$$

which implies

$$F\Big(\sum_{n=N}^{\infty}\frac{x^n}{n^\alpha}\Big)=\sum_{n=N}^{\infty}n^{-\alpha-\beta}.$$

However, for $\alpha>\frac{1}{2}$ and $\alpha+\beta\leqslant 1$ this equality is impossible. We note that we have then $\beta<\frac{1}{2}$. Moreover, we deduce that for every $\beta<\frac{1}{2}$ (and for $\alpha\in\big(\frac{1}{2},1-\beta\big]$) the continuous linear functional $F:\,L^2([0,1])\to\mathbb{R}$, such that the condition $F(x^n)=n^{-\beta}$ holds for almost all $n\in\mathbb{N}$, does not exist.

By reasoning similarly like in the proof of subitem $b)$ of Theorem 1 we can easily receive the following result.

Theorem 2. *Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Then we have*

$$\sum_{n=1}^{\infty}x^{p_n}\in L^2([0,1])\quad\Longleftrightarrow\quad\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{1}{p_k+p_n}<\infty.$$

Corollary 3. *If we additionally assume that the sequence $\{p_n\}_{n=1}^{\infty}$ is nondecreasing then we get*

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{1}{p_k+p_n}<\infty\quad\Longleftrightarrow\quad\sum_{n=1}^{\infty}\frac{n}{p_n}<\infty,$$

since the following estimation holds

$$\frac{n}{2\,p_n}\leqslant\sum_{k=1}^{n}\frac{1}{p_k+p_n}\leqslant\frac{n}{p_n},$$

for every $n\in\mathbb{N}$. Furthermore, if $\sum\limits_{n=1}^{\infty}\frac{n}{p_n^\gamma}=\infty$, for every $\gamma<1$, and

$$\sum_{n=1}^{\infty}\frac{x^{p_n}}{p_n^\alpha}\in L^2([0,1]),$$

for every $\alpha > \alpha_0$ *and for some* $\alpha_0 \in \left(0, \frac{1}{2}\right]$, *then a continuous linear functional* $F : L^2([0,1]) \to \mathbb{R}$ *such that* $F(x^{p_n}) = \frac{n}{p_n^\beta}$, *where* $\beta \in \mathbb{R}$, $\beta < 1 - \alpha_0$, *does not exist whenever* $\alpha + \beta < 1$.

2. Some connections with filters

We begin with the definition of a filter of subsets of $\mathbb{N}$ [1].

Definition 4. *A family* $\mathfrak{F}$ *of subsets of* $\mathbb{N}$ *is a filter on* $\mathbb{N}$ *if the following conditions hold true:*

(1) If $A \in \mathfrak{F}$ *and* $A \subset B \subset \mathbb{N}$ *then* $B \in \mathfrak{F}$.

(2) If $A, B \in \mathfrak{F}$ *then* $A \cap B \in \mathfrak{F}$.

We need a notion of the summable filter on $\mathbb{N}$.

Definition 5. *We say that a filter* $\mathfrak{F}$ *on* $\mathbb{N}$ *is a summable filter on* $\mathbb{N}$ *if there exists a sequence* $b = \{b_n\}_{n=1}^{\infty}$ *of the nonnegative real numbers such that* $\sum_{n=1}^{\infty} b_n = \infty$ *and*

$$\mathfrak{F} = \Big\{B \subset \mathbb{N} : \quad \sum_{n \in B} b_n = \infty \quad \text{and} \quad \sum_{n \in \mathbb{N} \setminus B} b_n < \infty\Big\}.$$

We note that this definition is correct. Indeed, if $B, C \subset \mathbb{N}$ and

$$\sum_{n \in B} b_n = \sum_{n \in C} b_n = \infty,$$

$$\sum_{n \in \mathbb{N} \setminus B} b_n < \infty \quad \text{and} \quad \sum_{n \in \mathbb{N} \setminus C} b_n < \infty,$$

then $\sum_{n \in B \cap C} b_n = \infty$, since $B = B \cap C \cup (B \setminus C)$ and $B \setminus C \subset \mathbb{N} \setminus C$. Moreover,

$$\sum_{n \in \mathbb{N} \setminus (B \cap C)} b_n < \infty \quad \text{since} \quad \mathbb{N} \setminus (B \cap C) = (\mathbb{N} \setminus B) \cup (\mathbb{N} \setminus C).$$

For a given $\alpha \in (0, 1]$ let us define

$$\mathfrak{F}_\alpha := \Big\{B \subset \mathbb{N} : \quad \sum_{n \in B} n^{-\alpha} = \infty \quad \text{and} \quad \sum_{n \in \mathbb{N} \setminus B} n^{-\alpha} < \infty\Big\}.$$

Filter $\mathfrak{F}_\alpha$ will be called the harmonic filter of order α. Let us notice that from the proof of Theorem 1 together with some simple arguments after this proof, the following result implies.

Theorem 6. *(a) If $B \subset \mathbb{N}$ then $\sum_{n\in B} \frac{x^n}{n^\alpha} \in L^2([0,1])$ for every $\alpha > \frac{1}{2}$.*

(b) If $\alpha \in (0,1]$ and $B \in \mathfrak{F}_\alpha$ then $\sum_{n\in B} \frac{x^n}{n^\beta} \in L^2([0,1])$ iff $\beta > \frac{1}{2}$.

(c) Furthermore, if $\alpha \in (0,1]$, $\beta \in \mathbb{R}$, $B \subset \mathbb{N}$ and $\sum_{n\in B} n^{-\alpha-\beta} = \infty$ then there is no continuous linear functional $F : L^2([0,1]) \to \mathbb{R}$ such that $F(x^n) = n^{-\beta}$, for all $n \in B$.

Let us present the final remarks.

Remark 7. If $\mathfrak{F}^*_\alpha$ is an ultrafilter containing $\mathfrak{F}_\alpha$ then, for every $B \in \mathfrak{F}^*_\alpha \setminus \mathfrak{F}_\alpha$, we have

$$\sum_{n\in B} n^{-\alpha} = \infty \quad \text{and} \quad \sum_{n\in \mathbb{N}\setminus B} n^{-\alpha} = \infty.$$

If either $B = 2\mathbb{N}$ or $B = 2\mathbb{N} - 1$ then

$$\sum_{n\in B} \frac{x^n}{n^\alpha} \in L^2([0,1]) \quad \Longleftrightarrow \quad \alpha > \frac{1}{2}. \tag{1}$$

Simultaneously, neither $2\mathbb{N}$ nor $2\mathbb{N} - 1$ do not belong to $\mathfrak{F}_\alpha$ for any $\alpha \in (0,1]$. However, the equality (1) is not a typical property for any pair $(B, \mathbb{N} \setminus B)$ of subsets of $\mathbb{N}$, such that neither B nor $\mathbb{N} \setminus B$ do not belong to $\mathfrak{F}_\alpha$ for any $\alpha \in (0,1]$. Nevertheless, this fact and some other problems will be discussed in the second part of this paper.

References

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Omówienie

W artykule omawiany jest problem istnienia ciągłego funkcjonału liniowego $F : L^2([0,1]) \to \mathbb{R}$, spełniającego warunek $F(x^n) = n^{-\alpha}$ dla prawie wszystkich $n \in \mathbb{N}$, gdzie α jest ustaloną liczbą dodatnią. Udowodniono, że jeśli $\alpha < \frac{1}{2}$, to taki funkcjonał nie istnieje. Przyjęta metoda dowodzenia pozwala znacząco uogólnić ten wynik. Przedstawia też sposób rozszerzenia otrzymanych twierdzeń z wykorzystaniem pojęcia filtru podzbiorów $\mathbb{N}$, zwłaszcza tak zwanych filtrów harmonicznych rzędu α. Okazuje się, że ultrafiltry zawierające filtry harmoniczne rzędu α generują ciekawe problemy natury analitycznej, związane z podziałem danych szeregów, których sumy należą do $L^2([0,1])$. Te oraz inne zagadnienia aproksymacyjne będą tematem naszych badań w kolejnych częściach niniejszego artykułu.