Determinants of the block arrowhead matrices

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Abstract. The paper is devoted to the considerations on determinants of the block arrowhead matrices. At first we discuss, in the wide technical aspect, the motivation of undertaking these investigations. Next we present the main theorem concerning the formulas describing the determinants of the block arrowhead matrices. We discuss also the application of these formulas by analyzing many specific examples. At the end of the paper we make an attempt to test the obtained formulas for the determinants of the block arrowhead matrices, but in case of replacing the standard inverses of the matrices by the Drazin inverses.

Keywords: determinant, block matrix, arrowhead matrix, Drazin inverse.

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1. Technical introduction – motivation for discussion

Observing the mathematical models, formulated for describing various dynamical systems used in the present engineering, one can notice that in many of them the block arrowhead matrices occur. One can find such matrices in the models of telecommunication systems [10], in robotics [18], in electrotechnics [17] and in automatic control [21]. In robotics, while modelling the dynamics of the kinematic chains of robots, or in electrotechnical problems, while modelling the electromechanical converters, there is often a need to formulate some equations in the coordinate systems different the ones in which the original model was formulated. Especially in the theory

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of electromechanical converters one applies very widely the possibility of transforming the mathematical models of the electromechanical converters form the natural coordinate systems into the biaxial coordinate systems. The main goal of these transformations is to eliminate the time dependence occurring in the coefficients of mutual inductances, to increase the number of zero elements occurring in the matrices of the appropriate mathematical model and, in consequence, to simplify this model and to shorten the time needed for its solution. The forms of matrices transforming the variables, occurring in the electromechanical converters, from the natural coordinate systems into the biaxial coordinate systems can be deduced on the way of physical reasoning. These matrices can be also found by using the methods of determining the eigenvalues. Formulated matrices are applied, among others in the Park, Clark and Stanley transformations [12, 9]. Elements of the transformation matrices contain the eigenvalues of matrices of the electromechanical converter inductance coefficients. These matrices can have, for example, the following forms [7]:

$$[K_s] = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos\vartheta_{s1} & \cos\vartheta_{s2} & \cos\vartheta_{s3} \\ \sin\vartheta_{s1} & \sin\vartheta_{s2} & \sin\vartheta_{s3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \tag{1}$$

$$[K_r] = \sqrt{\frac{2}{Q_r}} \begin{bmatrix} 1 & \cos\alpha_r & \cos 2\alpha_r & \ldots & \cos(Q_r-1)\alpha_r \\ 0 & -\sin\alpha_r & -\sin 2\alpha_r & \ldots & -\sin(Q_r-1)\alpha_r \\ 1 & \cos 2\alpha_r & \cos 4\alpha_r & \ldots & \cos 2(Q_r-1)\alpha_r \\ 0 & -\sin 2\alpha_r & -\sin 4\alpha_r & \ldots & -\sin 2(Q_r-1)\alpha_r \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \tag{2}$$

where

$$\vartheta_{s1} = \vartheta_{s10} + \int_0^t \Omega_x \, dt, \tag{3}$$

$$\vartheta_{s2} = \vartheta_{s20} + \int_0^t \Omega_x \, dt = \vartheta_{s10} + \frac{2\pi}{3} + \int_0^t \Omega_x \, dt, \tag{4}$$

$$\vartheta_{s3} = \vartheta_{s30} + \int_0^t \Omega_x \, dt = \vartheta_{s10} + \frac{4\pi}{3} + \int_0^t \Omega_x \, dt, \tag{5}$$

whereas ϑ_{s10}, ϑ_{s20}, ϑ_{s30} denote the initial angles between the axes of respective phases of the stator and axis X of the biaxial system XY for moment of time $t = 0$; p means the number of pole pairs, α_r denotes the rotor bar pitch and Ω_x describes the angular velocity of rotation of the biaxial system XY around the stator.

Matrix of the mutual and self inductance coefficients written in the natural coordinate system, that is in the phase system, possesses the block structure – it is the full matrix of the form

$$M = \begin{bmatrix} M_{ss} & M_{sr} \\ M_{sr}^T & M_{rr} \end{bmatrix}. \tag{6}$$

Particular forms of matrices, of this type can be found, for example, in [7].

Method of transforming the matrix (6) of inductance coefficients into the new coordinate system XY by using the transformation matrices (1) and (2) is as follows

$$\left[M_{ss}^{XY}\right] = \left[K_s\right]\left[M_{ss}\right]\left[K_s\right]^T, \tag{7}$$

$$\left[M_{rr}^{XY}\right] = \left[K_r\right]\left[M_{rr}\right]\left[K_r\right]^T, \tag{8}$$

$$\left[M_{sr}^{XY}\right] = \left[K_s\right]\left[M_{sr}\right]\left[K_r\right]^T. \tag{9}$$

By merging the obtained results we get the block arrowhead matrix [8], also in the form presented in paper [17].

Matrices (1) and (2) are often used for constructing the systems for regulating the electromechanical converters, such as the various versions of the vector control methods for the squirrel cage induction motors [13, 19]. Thus, there is a need for developing the effective methods of calculating the determinants of the block arrowhead matrices. Only to this issue the second part of this paper (see [17]) will be devoted.

2. The block arrowhead matrices

We use the following notation in this paper:

$\mathbb{0}$ denotes the zero matrix of the respective size (of dimensions $m \times n$),

$\mathbb{1}$ denotes the identity matrix of the respective order (of order n).

We present now the main result of this paper concerning the form of determinants of the block arrowhead matrices. In Section 3 we examine few examples illustrating the application of the obtained relations.

Theorem 2.1.

1. *Let us consider the block matrix $M_2 = \begin{bmatrix} A_{a \times a} & B_{a \times b} \\ C_{b \times a} & D_{b \times b} \end{bmatrix}$. The following implications hold:*

   (a) *If $\det D \neq 0$, then $\det M_2 = \det D \det(A - BD^{-1}C)$.*
   (b) *If $\det A \neq 0$, then $\det M_2 = \det A \det(D - CA^{-1}B)$.*
   In Remark 2.2, after the proof of the above theorem, we discuss the omitted situation when $\det A = \det D = 0$ and $a \neq b$.

2. *Let us consider the block matrix $M_3 = \begin{bmatrix} A_{a \times a} & B_{a \times b} & E_{a \times c} \\ C_{b \times a} & D_{b \times b} & \mathbb{0}_{b \times c} \\ F_{c \times a} & \mathbb{0}_{c \times b} & G_{c \times c} \end{bmatrix}$. The following implications hold*

   (a) *If $\det A \neq 0$ and $\det(D - CA^{-1}B) \neq 0$, then*

   $$\det M_3 = \det A \det(D - CA^{-1}B) \det(G - FA^{-1}E - \\ - FA^{-1}B(D - CA^{-1}B)^{-1}CA^{-1}E). \tag{10}$$

We note that if also $\det D \neq 0$, *then for the inverse of* $D - CA^{-1}B$ *the following Banachiewicz formula (see [1]), also called the Sherman-Morrison-Woodbury (SMW) formula (for some more historical information see [22, subchapter 0.8]), could be applied (see also S. Jose, K. C. Sivakumar, Moore-Penrose inverse of perturbated operators on Hilbert Spaces, 119-131, in [2]):*

$$(D - CA^{-1}B)^{-1} = D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1}.$$

The matrix $A - BD^{-1}C$ *is of order* $a \times a$ *and the above formula is useful in situations when* a *is much smaller than* b *and in all other situations when certain structural properties of* D *are much more simple than of* $D - CA^{-1}B$.

(b) *If* $\det D \neq 0$ *and* $\det G \neq 0$, *then*

$$\det M_3 = \det D \det G \det(A - BD^{-1}C - EG^{-1}F). \tag{11}$$

(c) *If either* $\mathrm{rank}\begin{bmatrix} B \\ D \end{bmatrix} + \mathrm{rank}\begin{bmatrix} E \\ G \end{bmatrix} \leq b + c - 1$ *or* $\mathrm{rank}\begin{bmatrix} C & D \end{bmatrix} + \mathrm{rank}\begin{bmatrix} F & G \end{bmatrix} \leq b + c - 1$, *then* $\det M_3 = 0$.

(d) *If* $\det G \neq 0$, $\det(A - EG^{-1}F) \neq 0$ *and* $D = \mathbb{0}$, *then*

$$\det M_3 = \det G \det(A - EG^{-1}F) \det(-C(A - EG^{-1}F)^{-1}B). \tag{12}$$

In the sequel, if blocks B *and* C *of* M_3 *are the square matrices (that is* $a = b$) *and* $D = \mathbb{0}$, *then*

$$\det M_3 = (-1)^a \det G \det B \det C. \tag{13}$$

At last, if

$$\mathrm{rank}\begin{bmatrix} A - EG^{-1}F & B \\ C & \mathbb{0} \end{bmatrix} = \mathrm{rank}\begin{bmatrix} A - EG^{-1}F \\ C \end{bmatrix} + \mathrm{rank}\begin{bmatrix} B \end{bmatrix}$$

$$= \mathrm{rank}\begin{bmatrix} C \end{bmatrix} + \mathrm{rank}\begin{bmatrix} A - EG^{-1}F & B \end{bmatrix} < a + b,$$

then $\det M_3 = 0$.

3. *The determinant of the following arrowhead matrix*

$$M_n = \begin{bmatrix} (A_{11})_{b_1 \times b_1} & (A_{12})_{b_1 \times b_2} & (A_{13})_{b_1 \times b_3} & \dots & (A_{1n})_{b_1 \times b_n} \\ (A_{21})_{b_2 \times b_1} & (A_{22})_{b_2 \times b_2} & \mathbb{0}_{b_2 \times b_3} & \dots & \mathbb{0}_{b_2 \times b_n} \\ (A_{31})_{b_3 \times b_1} & \mathbb{0}_{b_3 \times b_2} & (A_{33})_{b_3 \times b_3} & \dots & \mathbb{0}_{b_3 \times b_n} \\ \vdots & & & & \\ (A_{n1})_{b_n \times b_1} & \mathbb{0}_{b_n \times b_2} & \mathbb{0}_{b_n \times b_3} & \dots & (A_{nn})_{b_n \times b_n} \end{bmatrix} \tag{14}$$

is equal to

(a)

$$\det M_n = \det\left(A_{11} - \sum_{i=2}^{n} A_{1i}A_{ii}^{-1}A_{i1}\right)\prod_{j=2}^{n} \det A_{jj}, \tag{15}$$

whenever $\det A_{ii} \neq 0$ *for* $i = 2, 3, \dots, n$,

(b)

$$\det M_n = (-1)^{b_i} \det A_{i1} \det A_{1i} \prod_{\substack{j=2 \\ j \neq i}}^{n} \det A_{jj}, \qquad (16)$$

whenever there exists $i \in \{2, 3, \ldots, n\}$ such that $A_{ii} = \mathbb{0}$ and $b_i = b_1$.

4. The determinant of arrowhead matrix (14) is equal to

$$\det M_n = \det A_{11} \prod_{i=2}^{n} \det \left(A_{ii} - A_{i1} \left(A_{11} - \sum_{j=2}^{i-1} A_{1j} A_{jj}^{-1} A_{j1} \right)^{-1} A_{1i} \right), \qquad (17)$$

whenever $\det A_{jj} \neq 0$ for each $j = 1, 2, \ldots, n-1$ and $\det \left(A_{11} - \sum_{j=2}^{i-1} A_{1j} A_{jj}^{-1} A_{j1} \right) \neq 0$ for each $i = 3, 4, \ldots, n-1$. In case of $i = 2$ we take

$$\sum_{j=2}^{i-2} A_{1j} A_{jj}^{-1} A_{j1} := \mathbb{0}.$$

Proof.

1(a). Let us consider the case when D is the non-singular matrix. Then we have the following decomposition

$$M_2 = \begin{bmatrix} A_{a \times a} & B_{a \times b} \\ C_{b \times a} & D_{b \times b} \end{bmatrix} = \begin{bmatrix} \mathbb{1}_{a \times a} & \mathbb{0}_{a \times b} \\ \mathbb{0}_{b \times a} & D_{b \times b} \end{bmatrix} \begin{bmatrix} A_{a \times a} & B_{a \times b} \\ (D^{-1}C)_{b \times a} & \mathbb{1}_{b \times b} \end{bmatrix}.$$

Thus we get $\det M_2 = \det D \det \begin{bmatrix} A_{a \times a} & B_{a \times b} \\ (D^{-1}C)_{b \times a} & \mathbb{1}_{b \times b} \end{bmatrix}$. Next we can write

$$\begin{bmatrix} A_{a \times a} & B_{a \times b} \\ (D^{-1}C)_{b \times a} & \mathbb{1}_{b \times b} \end{bmatrix} = \begin{bmatrix} \mathbb{1}_{a \times a} & B_{a \times b} \\ \mathbb{0}_{b \times a} & \mathbb{1}_{b \times b} \end{bmatrix} \begin{bmatrix} (A - BD^{-1}C)_{a \times a} & \mathbb{0}_{a \times b} \\ (D^{-1}C)_{b \times a} & \mathbb{1}_{b \times b} \end{bmatrix},$$

from which we obtain

$$\det M_2 = \det D \det(A - BD^{-1}C).$$

1(b). Let us now assume that A is the non-singular matrix. Then the following equality holds true

$$\begin{bmatrix} \mathbb{1}_{a \times a} & \mathbb{0}_{a \times b} \\ (-CA^{-1})_{b \times a} & \mathbb{1}_{b \times b} \end{bmatrix} \begin{bmatrix} A_{a \times a} & B_{a \times b} \\ C_{b \times a} & D_{b \times b} \end{bmatrix} = \begin{bmatrix} A_{a \times a} & B_{a \times b} \\ \mathbb{0}_{b \times a} & (D - CA^{-1}B)_{b \times b} \end{bmatrix},$$

Since

$$\det \begin{bmatrix} \mathbb{1}_{a \times a} & \mathbb{0}_{a \times b} \\ (-CA^{-1})_{b \times a} & \mathbb{1}_{b \times b} \end{bmatrix} = 1,$$

thus we receive

$$\det \begin{bmatrix} A_{a\times a} & B_{a\times b} \\ C_{b\times a} & D_{b\times b} \end{bmatrix} = \det \begin{bmatrix} A_{a\times a} & B_{a\times b} \\ \mathbb{0}_{b\times a} & (D - CA^{-1}B)_{b\times b} \end{bmatrix} = \det A \det(D - CA^{-1}B).$$

2(a). Let $\det A \neq 0$ and $\det(D - CA^{-1}B) \neq 0$. Then we get the decomposition

$$\begin{bmatrix} \mathbb{1}_{a\times a} & \mathbb{0}_{a\times b} & \mathbb{0}_{a\times c} \\ -CA_{b\times a}^{-1} & \mathbb{1}_{b\times b} & \mathbb{0}_{b\times c} \\ U_{c\times a} & V_{c\times b} & \mathbb{1}_{c\times c} \end{bmatrix} \begin{bmatrix} A_{a\times a} & B_{a\times b} & E_{a\times c} \\ C_{b\times a} & D_{b\times b} & \mathbb{0}_{b\times c} \\ F_{c\times a} & \mathbb{0}_{c\times b} & G_{c\times c} \end{bmatrix} =$$

$$= \begin{bmatrix} A_{a\times a} & B_{a\times b} & E_{a\times c} \\ \mathbb{0}_{b\times a} & (D - CA^{-1}B)_{b\times b} & (-CA^{-1}E)_{b\times c} \\ \mathbb{0}_{c\times a} & \mathbb{0}_{c\times b} & W_{c\times c} \end{bmatrix},$$

where

$$U := -VCA^{-1} - FA^{-1},$$
$$V := FA^{-1}B(D - CA^{-1}B)^{-1},$$
$$W := G + UE = G - FA^{-1}E - VCA^{-1}E,$$

which implies formula (10).

2(b). Now let $\det D \neq 0$ i $\det G \neq 0$. Then we get the decomposition

$$\begin{bmatrix} A_{a\times a} & B_{a\times b} & E_{a\times c} \\ C_{b\times a} & D_{b\times b} & \mathbb{0}_{b\times c} \\ F_{c\times a} & \mathbb{0}_{c\times b} & G_{c\times c} \end{bmatrix} \begin{bmatrix} \mathbb{1}_{a\times a} & \mathbb{0}_{a\times b} & \mathbb{0}_{a\times c} \\ (-D^{-1}C)_{b\times a} & \mathbb{1}_{b\times b} & \mathbb{0}_{b\times c} \\ (-G^{-1}F)_{c\times a} & \mathbb{0}_{c\times b} & \mathbb{1}_{c\times c} \end{bmatrix} =$$

$$= \begin{bmatrix} (A - BD^{-1}C - EG^{-1}F)_{a\times a} & B_{a\times b} & E_{a\times c} \\ \mathbb{0}_{b\times a} & D_{b\times b} & \mathbb{0}_{b\times c} \\ \mathbb{0}_{c\times a} & \mathbb{0}_{c\times b} & G_{c\times c} \end{bmatrix},$$

which implies formula (11).

2(c). It is a folklore.

2(d). The following decomposition can be easily verified

$$\begin{bmatrix} A & B & E \\ C & \mathbb{0} & \mathbb{0} \\ F & \mathbb{0} & G \end{bmatrix} \begin{bmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{1} & \mathbb{0} \\ -G^{-1}F & \mathbb{0} & \mathbb{1} \end{bmatrix} = \begin{bmatrix} A - EG^{-1}F & B & E \\ C & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & G \end{bmatrix}.$$

Hence we obtain $\det M_3 = \det G \det \begin{bmatrix} A - EG^{-1}F & B \\ C & \mathbb{0} \end{bmatrix}$. Now the formula (12) follows from 1(b).

If blocks B and C of M_3 are the square matrices (that is $a = b$) and $D = \mathbb{0}$ then we obtain the decomposition given below

$$\begin{bmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{1} \\ \mathbb{0} & \mathbb{1} & \mathbb{0} \end{bmatrix} \begin{bmatrix} A & B & E \\ C & \mathbb{0} & \mathbb{0} \\ F & \mathbb{0} & G \end{bmatrix} \begin{bmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{1} \\ \mathbb{0} & \mathbb{1} & \mathbb{0} \end{bmatrix} = \begin{bmatrix} A & E & B \\ F & G & \mathbb{0} \\ C & \mathbb{0} & \mathbb{0} \end{bmatrix},$$
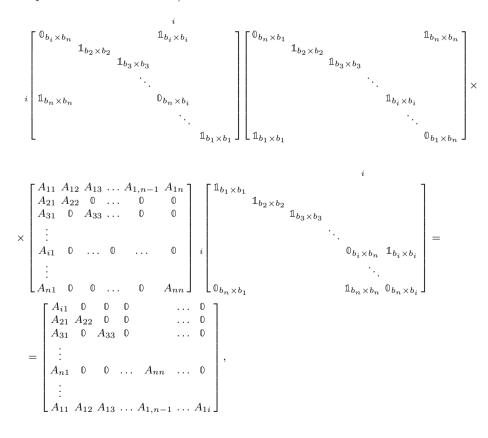
which implies formula (13).

3(a). Now let the matrices A_{ii}, for $i = 2, 3, \ldots, n$, be non-singular. Then we get the following decomposition

$$
\begin{bmatrix} A_{11} & A_{12} & A_{13} & \ldots & A_{1n} \\ A_{21} & A_{22} & \mathbb{0} & \ldots & \mathbb{0} \\ A_{31} & \mathbb{0} & A_{33} & \ldots & \mathbb{0} \\ \vdots & & & & \\ A_{n1} & \mathbb{0} & \mathbb{0} & \ldots & A_{nn} \end{bmatrix} \begin{bmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} & \ldots & \mathbb{0} \\ -A_{22}^{-1}A_{21} & \mathbb{1} & \mathbb{0} & \ldots & \mathbb{0} \\ -A_{33}^{-1}A_{31} & \mathbb{0} & \mathbb{1} & \ldots & \mathbb{0} \\ \vdots & & & & \\ -A_{nn}^{-1}A_{n1} & \mathbb{0} & \mathbb{0} & \ldots & \mathbb{1} \end{bmatrix} =
$$

$$
= \begin{bmatrix} A_{11} - \sum_{i=2}^{n} A_{1i}A_{ii}^{-1}A_{i1} & A_{12} & A_{13} & \ldots & A_{1n} \\ \mathbb{0} & A_{22} & \mathbb{0} & \ldots & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & A_{33} & \ldots & \mathbb{0} \\ \vdots & & & & \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & \ldots & A_{nn} \end{bmatrix}
$$

implying formula (15).

3(b). Now let $A_{ii} = \mathbb{0}$ for some $i \in \{2, 3, \ldots, n\}$ and let $b_i = b_1$. We obtain the following decomposition (where on the left hand side of the equality sign we have the product of four matrices):

$$
i \begin{bmatrix} \mathbb{0}_{b_i \times b_n} & & & & \mathbb{1}_{b_i \times b_i} & & \\ & \mathbb{1}_{b_2 \times b_2} & & & & & \\ & & \mathbb{1}_{b_3 \times b_3} & & & & \\ & & & \ddots & & & \\ \mathbb{1}_{b_n \times b_n} & & & & \mathbb{0}_{b_n \times b_i} & & \\ & & & & & \ddots & \\ & & & & & & \mathbb{1}_{b_1 \times b_1} \end{bmatrix} \begin{bmatrix} \mathbb{0}_{b_n \times b_1} & & & & \mathbb{1}_{b_n \times b_n} \\ & \mathbb{1}_{b_2 \times b_2} & & & \\ & & \mathbb{1}_{b_3 \times b_3} & & \\ & & & \ddots & \\ & & & \mathbb{1}_{b_i \times b_i} & \\ & & & & \ddots \\ \mathbb{1}_{b_1 \times b_1} & & & & \mathbb{0}_{b_1 \times b_n} \end{bmatrix} \times
$$

$$
\times \begin{bmatrix} A_{11} & A_{12} & A_{13} & \ldots & A_{1,n-1} & A_{1n} \\ A_{21} & A_{22} & \mathbb{0} & \ldots & \mathbb{0} & \mathbb{0} \\ A_{31} & \mathbb{0} & A_{33} & \ldots & \mathbb{0} & \mathbb{0} \\ \vdots & & & & & \\ A_{i1} & \mathbb{0} & \ldots & \mathbb{0} & \ldots & \mathbb{0} \\ \vdots & & & & & \\ A_{n1} & \mathbb{0} & \mathbb{0} & \ldots & \mathbb{0} & A_{nn} \end{bmatrix} \begin{bmatrix} \mathbb{1}_{b_1 \times b_1} & & & & \\ & \mathbb{1}_{b_2 \times b_2} & & & \\ & & \mathbb{1}_{b_3 \times b_3} & & \\ & & & \ddots & \\ & & & \mathbb{0}_{b_i \times b_n} & \mathbb{1}_{b_i \times b_i} \\ & & & & \ddots \\ \mathbb{0}_{b_n \times b_1} & & & \mathbb{1}_{b_n \times b_n} & \mathbb{0}_{b_n \times b_i} \end{bmatrix} =
$$

$$
= \begin{bmatrix} A_{i1} & \mathbb{0} & \mathbb{0} & \mathbb{0} & & \ldots & \mathbb{0} \\ A_{21} & A_{22} & \mathbb{0} & \mathbb{0} & & \ldots & \mathbb{0} \\ A_{31} & \mathbb{0} & A_{33} & \mathbb{0} & & \ldots & \mathbb{0} \\ \vdots & & & & & & \\ A_{n1} & \mathbb{0} & \mathbb{0} & \ldots & A_{nn} & \ldots & \mathbb{0} \\ \vdots & & & & & & \\ A_{11} & A_{12} & A_{13} & \ldots & A_{1,n-1} & \ldots & A_{1i} \end{bmatrix},
$$

from which we receive

$$\det M_n = \alpha \det A_{i1} \det A_{1i} \prod_{\substack{j=2 \\ j \neq i}}^{n} \det A_{jj},$$

where

$$\alpha = (-1)^{b_n \sum\limits_{k=2}^{i} b_k + b_i \sum\limits_{k=2}^{i} b_k} \cdot (-1)^{b_1 \sum\limits_{k=2}^{n} b_k + b_n \sum\limits_{k=2}^{n} b_k} \cdot (-1)^{b_n \sum\limits_{k=i}^{n} b_k + b_i \sum\limits_{k=i}^{n-1} b_k} =$$

$$= (-1)^{2b_n \sum\limits_{k=2}^{n} b_k + b_i \sum\limits_{k=2}^{n} b_k + b_i^2 + b_1 \sum\limits_{k=2}^{n} b_k} = (-1)^{2b_1 \sum\limits_{k=2}^{n} b_k + b_i^2} = (-1)^{b_i^2} = (-1)^{b_i}.$$

4. Let X denote the block matrix given below

$$X = \begin{bmatrix} \mathbb{1} & & & & \mathbb{0} \\ X_{21} & \mathbb{1} & & & \\ X_{31} & X_{22} & \mathbb{1} & & \\ \vdots & \vdots & \vdots & \ddots & \\ X_{n1} & X_{n2} & X_{n3} & \ldots & \mathbb{1} \end{bmatrix},$$

where

$$X_{ij} = \begin{cases} -A_{i1} \left(A_{11} - \sum\limits_{k=2}^{i-1} A_{1k} A_{kk}^{-1} A_{k1} \right)^{-1}, & i > j = 1, \\ A_{i1} \left(A_{11} - \sum\limits_{k=2}^{i-1} A_{1k} A_{kk}^{-1} A_{k1} \right)^{-1} A_{1j} A_{jj}^{-1}, & i > j > 1. \end{cases}$$

Since this is the lower triangular matrix, therefore its determinant is equal to

$$\det X = (\det \mathbb{1})^n = 1.$$

Hence, of we introduce the notation $Y = X M_n$, then the quality holds true

$$\det M_n = \det X \det M_n = \det(X M_n) = \det Y.$$

Let us find matrix Y. If condition $i = j = 1$ is fulfilled, then we get

$$Y_{11} = \mathbb{1} A_{11} = A_{11}.$$

Next, if $i = j > 1$, then we obtain

$$Y_{ii} = -A_{i1} \left(A_{11} - \sum_{k=2}^{i-1} A_{1k} A_{kk}^{-1} A_{k1} \right)^{-1} A_{1i} + A_{ii} =$$

$$= A_{ii} - A_{i1} \left(A_{11} - \sum_{k=2}^{i-1} A_{1k} A_{kk}^{-1} A_{k1} \right)^{-1} A_{1i}.$$

In turn, if $i > j = 1$, then we have

$$Y_{i1} = -A_{i1}\Big(A_{11} - \sum_{k=2}^{i-1} A_{1k}A_{kk}^{-1}A_{k1}\Big)^{-1} A_{11} +$$

$$+ \sum_{l=2}^{i-1} A_{i1}\Big(A_{11} - \sum_{k=2}^{i-1} A_{1k}A_{kk}^{-1}A_{k1}\Big)^{-1} A_{1l}A_{ll}^{-1}A_{l1} + A_{i1} =$$

$$= -A_{i1}\Big(A_{11} - \sum_{k=2}^{i-1} A_{1k}A_{kk}^{-1}A_{k1}\Big)^{-1}\Big(A_{11} - \sum_{l=2}^{i-1} A_{1l}A_{ll}^{-1}A_{l1}\Big) + A_{i1} = \mathbb{0}.$$

And finally, for $i > j > 1$ we obtain

$$Y_{ij} = -A_{i1}\Big(A_{11} - \sum_{k=2}^{i-1} A_{1k}A_{kk}^{-1}A_{k1}\Big)^{-1} A_{1j} +$$

$$+ A_{i1}\Big(A_{11} - \sum_{k=2}^{i-1} A_{1k}A_{kk}^{-1}A_{k1}\Big)^{-1} A_{1j}A_{jj}^{-1}A_{jj} =$$

$$= -A_{i1}\Big(A_{11} - \sum_{k=2}^{i-1} A_{1k}A_{kk}^{-1}A_{k1}\Big)^{-1} A_{1j} + A_{i1}\Big(A_{11} - \sum_{k=2}^{i-1} A_{1k}A_{kk}^{-1}A_{k1}\Big)^{-1} A_{1j} = \mathbb{0},$$

from which we conclude that matrix Y is the upper triangular matrix. It means that its determinant is equal to

$$\det Y = \det A_{11} \prod_{i=2}^{n} \det\Big(A_{ii} - A_{i1}\Big(A_{11} - \sum_{k=2}^{i-1} A_{1k}A_{kk}^{-1}A_{k1}\Big)^{-1} A_{1i}\Big),$$

which leads directly, in view of the previous equalities, to formula (17).

$\square$

Remark 2.2. If

$$\det A = \det D = 0 \text{ and } a \neq b, \tag{18}$$

and despite of this $\det M_2 \neq 0$, then we propose to change the decomposition of matrix M_2 into blocks so that the "new" matrices A and D would satisfy relations $\det A \neq 0$ or $\det D \neq 0$.

Let us only notice that in case of fulfilling conditions (18), if $A = \mathbb{0}$ and $D = \mathbb{0}$, then

$$\operatorname{rank} M_2 = \operatorname{rank} C + \operatorname{rank} D \leq 2\min\{a, b\} < a + b,$$

that is $\det M_2 = 0$.

Hence, if $A \neq \mathbb{0}$, then, in the worst case, after the possible permutation of the respective rows and columns (or only the rows, or only the columns), one can get that $a_{11} \neq 0$ and take $A = [a_{11}]$.

For example, let $M_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. The new decomposition into blocks

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

solves this problem.

Let us consider additionally the following example:

$$M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 3 & 1 & 2 \\ 3 & 1 & 1 & 1 & 4 \\ 1 & 0 & 2 & 1 & 5 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} , \quad \det M_2 = 2.$$

Let us notice that in this example for any decomposition into blocks we always have $\det A = \det D = 0$. Thus, for the non-singularity of blocks A or D we need to permutate the rows (or the columns, alternatively). In the considered case, by replacing the first and third rows we receive

$$\begin{bmatrix} 3 & 1 & 1 & 1 & 4 \\ 2 & 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 1 & 5 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

Then the new block A is non-singular, thus, by applying item 1(b) from the above theorem, we can calculate the determinant of matrix M_2 (by keeping in mind that the permutation of rows causes the change of sign of the determinant).

3. Examples

Example 3.1. Example of a symmetric non-singular matrix, to which one cannot apply any of formulas (10), (11), (15) and (17):

$$B_1 = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} , \quad \det B_1 = 20.$$

Formulas (11), (15), (17) cannot be used because matrix B_1 contains the singular block $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Moreover, one cannot apply formula (10) because matrix

$$D - CA^{-1}B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

is singular.

Example 3.2. Example of a singular matrix, to which one can apply formulas (10), (11), (15), (17):

$$B_2 = \left[\begin{array}{cc|cc|cc} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right], \quad \det B_2 = 0.$$

Example 3.3. Example of a non-singular matrix, to which one can apply formulas (10), (11), (15), (17):

$$B_3 = \left[\begin{array}{cc|cc|cc} 3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right], \quad \det B_3 = 1.$$

Example 3.4. Example of a matrix with non-diagonal blocks, to which one can apply formulas (10), (11), (15), (17):

$$B_4 = \left[\begin{array}{cc|cc|cc} 1 & 2 & 2 & 7 & 1 & 2 \\ 2 & 5 & 1 & 4 & 3 & 4 \\ \hline 2 & 1 & 1 & 3 & 0 & 0 \\ 7 & 4 & 3 & 8 & 0 & 0 \\ \hline 1 & 3 & 0 & 0 & 2 & 1 \\ 2 & 4 & 0 & 0 & 1 & 3 \end{array} \right], \quad \det B_4 = -14.$$

Example 3.5. Example of the class of matrices, to which one can apply formula (17):

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & \ldots & A_{1n} \\ A_{21} & A_{22} & \mathbb{0} & \ldots & \mathbb{0} \\ A_{31} & \mathbb{0} & A_{33} & \ldots & \mathbb{0} \\ & & & \vdots & \\ A_{n1} & \mathbb{0} & \mathbb{0} & \ldots & A_{nn} \end{bmatrix},$$

where

- A_{11} is the diagonal matrix of dimensions $m \times m$ of positive elements,
- A_{kk}, for $k = 2, 3, \ldots, n-1$, are the diagonal matrices of dimensions $m \times m$ of negative elements,
- A_{1k}, for $k = 2, 3, \ldots, n$, are any diagonal matrices of dimensions $m \times m$ of non-zero elements,
- $A_{k1} = A_{1k}$.

Thus in formula

$$\det A_{11} \prod_{i=2}^{n} \det \left(A_{ii} - A_{i1} \left(A_{11} - \sum_{k=2}^{i-1} A_{1k} A_{kk}^{-1} A_{k1} \right)^{-1} A_{1i} \right)$$

we have as follows

- A_{kk}, for $k = 2, \ldots, n-1$, are non-singular since they are diagonal of negative elements,
- $A_{11} - \sum_{k=2}^{i-1} A_{1k} A_{kk}^{-1} A_{k1} = A_{11} - \sum_{k=2}^{i-1} A_{1k}^2 A_{kk}^{-1}$, for $i = 2, 3, \ldots, n$, are the diagonal matrices of positive elements, therefore they are non-singular.

Remark 3.6. Analogically one can take the diagonal matrix of negative elements as A_{11} and the diagonal matrices of positive elements as A_{kk} for $k = 2, 3, \ldots, n-1$.

Example 3.7. Example of a matrix with rectangular blocks, to which one can apply formulas (10), (11), (15) and (17):

$$B_5 = \left[\begin{array}{cc|c|ccc} 1 & 0 & 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 4 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 4 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 & 0 & 3 \end{array} \right] , \quad \det B_5 = 312.$$

Remark 3.8. Let us consider the following arrowhead matrix of dimensions $n \times n$ of complex elements

$$A_n = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & & & \\ a_{31} & & a_{33} & & \mathbf{0} \\ \vdots & & & \ddots & \\ a_{n1} & & \mathbf{0} & & a_{nn} \end{bmatrix} ,$$

where symbol $\mathbf{0}$ means that all the other respective elements of matrix A_n are equal to zero. One can show inductively that the determinant of matrix A_n is expressed by formula

$$\det A_n = \prod_{i=1}^{n} a_{ii} - \sum_{i=2}^{n} a_{1i} a_{i1} \prod_{\substack{j=2 \\ j \neq i}}^{n} a_{jj}, \tag{19}$$

where we define $\sum_{i=2}^{n} a_{1i} a_{i1} \prod_{\substack{j=2 \\ j \neq i}}^{n} a_{jj} = \begin{cases} 0 & \text{for } n = 1, \\ a_{12} a_{21} & \text{for } n = 2. \end{cases}$

One can easily notice that the formula (19) is analogue to formula (15) for the non-block arrowhead matrices, however, what should be especially emphasized, it does not require the assumption about non-zero elements a_{ii} for $i = 2, 3, \ldots, n$.

Example 3.9. Let us consider the determinant of matrix

$$
M = \left[\begin{array}{cccc|cccc}
1 & -1 & 1 & 1 & 2 & 1 & 2 & 1 \\
-1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 3 & 0 & 0 & 2 \\
\hline
5 & 9 & 0 & 0 & 1 & 0 & 0 & 0 \\
-17 & -\frac{5}{2} & -15 & 3 & 0 & 1 & 0 & 0 \\
6 & 0 & 9 & 0 & 0 & 0 & 1 & 0 \\
-4 & -\frac{27}{2} & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right].
$$

We denote the blocks of matrix M in the following way

$$
A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad
B = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}, \quad
C = \begin{bmatrix} 5 & 9 & 0 & 0 \\ -17 & -\frac{5}{2} & -15 & 3 \\ 6 & 0 & 9 & 0 \\ -4 & -\frac{27}{2} & 0 & 1 \end{bmatrix}, \quad
D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

The determinant of matrix M can be calculated by using Theorem 2.1 from item 1 (a) since $\det D = 1 \neq 0$. Thus we get

$$
\det M = \det D \det(A - BD^{-1}C) = \det(A - BC).
$$

We calculate the determinant of matrix

$$
A - BC = \begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 5 & 9 & 0 & 0 \\ -17 & -\frac{5}{2} & -15 & 3 \\ 6 & 0 & 9 & 0 \\ -4 & -\frac{27}{2} & 0 & 1 \end{bmatrix} =
$$

$$
= \begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 9 & 0 & 0 \\ 6 & 0 & 9 & 0 \\ 7 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -3 & -2 & -3 \\ -6 & -8 & 0 & 0 \\ -5 & 0 & -8 & 0 \\ -6 & 0 & 0 & -1 \end{bmatrix}.
$$

Let us observe that matrix $A - BC$ is the arrowhead matrix, therefore its determinant can be easily calculated by using formula (19). We obtain

$$
\det M = \det(A - BC) = \det \begin{bmatrix} 0 & -3 & -2 & -3 \\ -6 & -8 & 0 & 0 \\ -5 & 0 & -8 & 0 \\ -6 & 0 & 0 & -1 \end{bmatrix} =
$$

$$
= 0 - (18 \cdot 8 + 10 \cdot 8 + 18 \cdot 64) = -1376.
$$

4. Tests with the inverse matrices in the Drazin sense

The Drazin inverse of matrix $A \in \mathbb{C}^{n \times n}$ is defined to be the unique matrix $A^d \in \mathbb{C}^{n \times n}$ such that

1. $A^d A A^d = A^d$,
2. $A A^d = A^d A$,
3. $A^d A^{m+1} = A^m$,

where $m = \mathrm{ind}(A)$, which means the index of A, i.e. the smallest nonnegative integer m for which $\mathrm{rank}(A^m) = \mathrm{rank}(A^{m+1})$.

We want to try to compare the influence of using the Drazin inverse matrix in formula (17) on the value of determinant in case of matrices not satisfying the assumptions of Theorem 2.1.4.

Example 4.1. Example of a singular matrix, in case of which the application of the Drazin invertible matrices leads to the incorrect result

$$B_6 = \left[\begin{array}{cc|cc|cc} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right], \quad \det B_6 = 0.$$

If we want to use formula (17), we have to invert matrix $A_{22} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Applying the Drazin inverse matrix we obtain $A_{22}^d = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$ (we used here the respective algorithm for finding A^d given in [5, chapter 1.4.4, page 54]). Substituting this matrix into the formula in place of A_{22}^{-1} we get $\det B_6 = \frac{2}{5}$, so the result is incorrect.

Example 4.2. Example of a singular matrix, to which formula (17) cannot be applied, but when we use in the singular place the Drazin inverse matrix, we have the correct result

$$B_7 = \left[\begin{array}{cc|cc|cc} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right], \quad \det B_7 = 0.$$

In this case the singularity results from the form of the discussed formula. So we have $B = A_{11} - A_{12} A_{22}^{-1} A_{12}^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Calculating the Drazin inverse matrix we receive $B^d = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and next, by using it in the formula in place of $(A_{11} - A_{12} A_{22}^{-1} A_{12}^T)^{-1}$ we obtain the correct result.

Remark 4.3. In Example 4.1 the singularity occurs in the form of a singular block. In this case as well, the application of the Drazin inverse matrix leads to the incorrect result. In Example 4.2 the singularity results from the singularity of "a part of the formula", that is $A_{11} - A_{12}A_{22}^{-1}A_{12}^T$, and in this case the application of the Drazin inverse matrix gives the correct result. Some further discussion, especially the theoretical one, with the use of the Drazin matrices, we decided to perform in a separate paper.

Special supplement

After receiving the review of this paper we have discovered two more publications and we decided to include them into the references, because we find them as essentially connected with the investigated subject matter and really deserving to be noticed.

The first paper, by V. Katsnelson [6], concerns the application and, simultaneously, the additional interpretation of the Herbert's Stahl Theorem. In this paper, from among many technical masterpieces, one can find a polynomial of two variables, called as the polynomial pencil, being the characteristic polynomial of some explicitly given arrow matrix (called by the Author as the matrix pencil).

The second paper, written by N. Stojković and P. Stanimirovic [15], refers to the block matrices discussed also in our paper. Form of the determinants of the respective block matrices is derived in this paper on the basis of the determined characteristic polynomials of these matrices.

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