Infinite Generalizations of Hamming Spaces and their Isometry Groups

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Hamming Space

Denote by H_n the *Hamming space* of dimension n.
This space consists of all n-tuples

$$(a_1, \dots, a_n), \quad a_i \in \{0, 1\}, 1 \leq i \leq n.$$

The distance d_{H_n} between two such n-tuples is equal to the number of coordinates where they differ.

Hamming Space

The Hamming metric space H_n plays an essential role in computer science. The Hamming metric was introduced in coding theory as the most common tool to measure the difference between binary strings of the equal length.

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- From the metric geometry viewpoint the Hamming spaces H_n, $n \geq 1$, form a universal family among finite metric spaces with respect to isomorphic embeddings.

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- From the graph theoretical viewpoint the Hamming space H_n is the n-dimensional hypercube.
- From the metric geometry viewpoint the Hamming spaces H_n, $n \geq 1$, form a universal family among finite metric spaces with respect to isomorphic embeddings.
- From the group theoretical viewpoint the Hamming space H_n is the space whose isometries form the semidirect product

$$\bigoplus_{i=1}^{n} S_2^{(i)} \rtimes S_n.$$

Isometries of Hamming Space

In other words the isometry group $IsomH_n$ of the metric space H_n is isomorphic to the wreath product $W_n = S_2 \wr S_n$.

- A. Kerber. *Representations of permutation groups*, 1971.
- M.Ch. Klin, R. Póschel, K. Rosenbaum. *Angewandte Algebra fúr Mathematiker und Informatiker. Einfúhrung in gruppentheoretisch-kombinatorische Methoden. (Applied algebra for mathematicians and computer scientists. Introduction to group theoretical combinatorial methods)*, 1988.

Countable Hamming Space

Let $\{0,1\}^{\mathbb{N}}$ be the set of all infinite tuples of elements of the set $\{0,1\}$, i.e. the set of all infinite binary sequences.
The countable Hamming space ("Countable cube") $H_{\mathbb{N}}$ consists of all infinite tuples

$$(a_1, a_2, \dots), \quad a_i \in \{0,1\}, i \geq 1,$$

such that almost all their coordinates equal zero (i.e. only finite number of coordinates equal one).
The distance between two such infinite tuples is equal to the number of coordinates where they differ.

Isometries of Countable Hamming Space

Let $g_2 : \mathbb{N} \to S_2$, i.e. $g_2 \in S_2^{\mathbb{N}}$. Denote by $supp(g_2)$ the set of all elements $x_1 \in \mathbb{N}$, such that $g_2(x_1) \neq Id_{S_2}$. Define a restricted wreath product

$$S_2 \bar{\wr} S_{\mathbb{N}} = \{[g_2(x_1), g_1] \mid g_1 \in S_{\mathbb{N}}, \quad g_2(x_1) \in S_2^{\mathbb{N}}, \quad |supp(g_2)| < \infty\},$$

as a subgroup of $S_2 \wr S_{\mathbb{N}}$.

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as a subgroup of $S_2 \wr S_{\mathbb{N}}$.

Theorem (B. O. [1996], M. Pankov [2012])

The isometry group $Isom H_{\mathbb{N}}$ of the countable Hamming space $H_{\mathbb{N}}$ is isomorphic to the restricted wreath product

$$S_2 \bar{\wr} S_{\mathbb{N}}.$$

Extended Infinite Hamming Space

The extended infinite Hamming space $H_{\mathbb{R}}$ is the space of all infinite $\{0, 1\}$-sequences. The extended distance between two points is equal to the number of coordinates where they differ, if the number of such coordinates is finite, and equals ∞ otherwise.

Extended Infinite Hamming Space

The extended infinite Hamming space $H_{\mathbb{R}}$ is the space of all infinite $\{0,1\}$-sequences. The extended distance between two points is equal to the number of coordinates where they differ, if the number of such coordinates is finite, and equals ∞ otherwise.

Theorem (B.O. [2013])

The isometry group of the extended infinite Hamming space $H_{\mathbb{R}}$ is isomorphic to the wreath product $(S_2 \overline{\wr}\, S_{\mathbb{N}}) \wr S_{\mathbb{R}}$.

Besicovitch space

Let us now equip the set $\{0,1\}^{\mathbb{N}}$ with a pseudo-metric $\hat{d}_B$, defined for arbitrary sequences $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$ from $\{0,1\}^{\mathbb{N}}$ by the equality

$$\hat{d}_B(x, y) = \limsup_{n\to\infty} \frac{1}{n} d_{H_n}((x_1, \dots, x_n), (y_1, \dots, y_n)).$$

Besicovitch space

The pseudo-metric $\hat{d}_B$ defines an equivalence $\sim_{\hat{d}_B}$ on $\{0,1\}^{\mathbb{N}}$, i.e.

$$\mathbf{x} \sim_{\hat{d}_B} \mathbf{y}$$

if and only if

$$\hat{d}_B(\mathbf{x}, \mathbf{y}) = 0.$$

Denote by $X_B = \{0,1\}^{\mathbb{N}} / \sim_{\hat{d}_B}$ the quotient set by this equivalence. The function $\hat{d}_B$ induces the metric d_B on X_B.

Besicovitch space

The metric space (X_B, d_B) is called the Besicovitch space.
(see F. Blanchard , E. Formenti, P. Kurka, *Cellular Automata in Cantor, Besicovitch and Weil Topological Spaces*, Complex Systems, V. 11, 1997, pp. 107–123,
or A. M. Vershik, *The Pascal automorphism has a continuous spectrum*, Funct. Anal. Appl., V. 45, 2011, pp 173–186).

Besicovitch space

The metric space (X_B, d_B) is complete, nonseparable, and not locally compact.
We consider a continuum family of compact separable subspaces of the Besicovitch space, naturally parameterized by supernatural numbers.

Steinitz numbers

Let $\mathbb{P}$ be the set of all primes. A *supernatural number* (or Steinitz number) is an infinite formal product of the form

$$\prod_{p\in\mathbb{P}} p^{k_p}$$

where $k_p \in \mathbb{N} \cup \{0, \infty\}$. Denote by $\mathbb{SN}$ the set of all supernatural numbers. The elements of the set $\mathbb{SN} \setminus \mathbb{N}$ are called *infinite supernatural* numbers.

Periodic sequences

An infinite sequence $\mathbf{a} = (a_1, a_2, \dots)$ is said to be *periodic* if there exists a natural number k such that the equality

$$a_i = a_{i+k}$$

holds for all $i \in \mathbb{N}$. In this case the number k is called a *period* of the sequence $\mathbf{a}$.

A periodic sequence $\mathbf{a}$ is called *u-periodic* for some supernatural number u if its minimal period divides u.

Periodic Hamming spaces

Let u be some infinite supernatural number. Denote by $\mathcal{H}(u)$ the subspace of the Besicovitc space (X_B, d_B) consisting of all u-periodic sequences over the set $\{0, 1\}$. We call the metric space $\mathcal{H}(u)$ the *u-periodic Hamming space*.

Proposition

Let u, v be supernatural numbers. Then the spaces $\mathcal{H}(u)$ and $\mathcal{H}(v)$ are isometric if and only if $u = v$.

Besicovitch-Hamming space

Proposition

[P. J. Cameron, S. Tarzi, 2008]
The completions $\mathcal{H}$ of u-periodic Hamming spaces are independent of choice of u.

The completion $\mathcal{H}$ is called the *Besicovitch-Hamming* space.

Problem

[A. M. Vershik, 2012]
Let $(x_1, x_2, \ldots)$ be from $\{0,1\}^{\mathbb{N}}$ and let $\mathbf{x}$ be the equivalence class defined by the sequence $(x_1, x_2, \ldots)$. Is there an algorithm to determine whether a class $\mathbf{x}$ belongs to the Besicovitch-Hamming space $\mathcal{H}$?

Besicovitch-Hamming space

Theorem

[P. J. Cameron, S. Tarzi, 2008]
(a) The points of $H(2^\infty)$ can be identified with the subsets of [0, 1) which are unions of finitely many half-open intervals [x, y) with dyadic rational endpoints, the distance between two such sets being the sum of the lengths of their symmetric difference.
(b) The points of $\mathcal{H}$ can be identified with the Lebesgue measurable subsets of [0, 1] modulo null sets, the distance between two points being the Lebesgue measure of their symmetric difference.

Characteristics

A sequence of positive integers $\tau = (m_1, m_2, \ldots)$ is called *divisible* if $m_i | m_{i+1}$ for all $i \in \mathbb{N}$.
Let $\tau = (m_1, m_2, \ldots)$ be an increasing divisible sequence. Denote by $(s_1, s_2, \ldots)$ the sequence of ratios of the sequence τ, i.e.

$$s_1 = m_1, \qquad s_{i+1} = \frac{m_{i+1}}{m_i}, \ i \geq 1.$$

The supernatural number

$$s_1 \cdot s_2 \cdot s_3 \ldots$$

is called the *characteristic of the sequence* τ and denoted by $char(\tau)$.

Rooted Trees

Assume that T_τ is a spherically homogeneous rooted tree with spherical index $[s_1; s_2; \ldots]$. We consider the boundary ∂T_τ of the tree T_τ, i.e. the set of all infinite simple paths starting at the root. We call these paths rooted paths.

Path Metric

Define a distance ρ on the set ∂T_τ as

$$\rho_\tau(\gamma_1, \gamma_2) = \begin{cases} \frac{1}{k+1}, & \text{if } \gamma_1 \neq \gamma_2 \\ 0, & \text{if } \gamma_1 = \gamma_2 \end{cases},$$

where k is the length of the common beginning of rooted paths γ_1 and γ_2.

Path Metric Topology

Consider the topology on ∂T_τ induced by the metric ρ_τ. Finite unions of cylindrical sets are open (and closed) sets in this topology. The set of all rooted paths from ∂T_τ passing through a vertex v is denoted by

$$C_v = \{\gamma \in \partial T_\tau \ \mid \ v \in \gamma\}$$

and called the *cylindrical set* C_v corresponding to v.

Bernoulli Measure

Define the Bernoulli measure μ on the Borel σ-algebra of clopen sets of ∂T_τ using the rule:

$$\mu(C_v) = \frac{1}{n_v},$$

where n_v is the number of vertices of T_τ on the level containing the vertex v.

Periodic Hamming Spaces and Rooted Trees

Define the metric d_μ on the set ΩT_τ of all clopen subsets of ∂T_τ by putting $d_\mu(A, B) = \mu(A \triangle B)$ for all clopen subsets A and B of ∂T_τ.

Theorem (B.O., V. Sushchansky [2013])

Let $\tau = (m_1, m_2, \ldots)$ be a strictly increasing divisible sequence of positive integers with the sequence $[s_1; s_2; \ldots,]$ of its ratios, T_τ be the spherically homogeneous rooted tree by type $[s_1; s_2; \ldots,]$. Assume that u is a supernatural number such that $char(\tau) = u$. Then the Hamming space $H(u)$ of all u-periodic $(0, 1)$-sequences is isometric to the space ΩT_τ of all clopen subsets of ∂T_τ equipped with the metric d_μ.

Besicovitch-Hamming Space and Rooted Trees

Corollary (B.O., V. Sushchansky [2013])

The Besicovitch-Hamming space $\mathcal{H}$ is isometric to the space of all measurable subsets (up to measure zero sets) of ∂T_τ equipped with the metric d_μ.

Spherically Transitive Automorphisms

An automorphism u of spherically homogeneous rooted tree T_τ is called spherically transitive, if the cyclic group $\langle u \rangle$ acts transitively on each level of the tree T_τ. A typical example of a spherically transitive automorphism is the "adding machine".

Adding Machine

The adding machine is an automorphism of the spherically homogeneous rooted tree T_υ with spherical index $\nu = [n; n; \ldots,]$. This automorphism can be defined via the following figure:

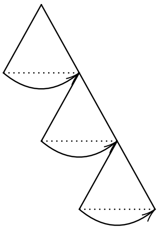

Figure 1:

Spherically Transitive Automorphisms

Consider arbitrary spherically transitive automorphism $w \in AutT_\tau$. Each automorphism α of the rooted tree T_τ acts as an isometry on the boundary $(\partial T_\tau, \rho)$ and vise versa. Therefore w can be considered as an isometry of ∂T_τ.

Construction of Isometry

Let t_0 be a fixed point from the boundary ∂T_τ. For any subset $A \subset \partial T_\tau$ let us define an infinite $(0,1)$-sequence $s_w(A) = (a_0, a_1, a_2, \ldots)$ by the rule

$$a_n = \begin{cases} 1, & \text{if } w^n(t_0) \in A \\ 0, & \text{if } w^n(t_0) \notin A \end{cases}. \quad (1)$$

In this way we obtain the mapping F_w defined on the set of all subsets of the boundary ∂T_τ to the set of all infinite $(0,1)$-sequences. Let f_w denotes the restriction of F_w on the set ΩT_τ of all clopen subsets of the boundary ∂T_τ.

Periodic Hamming Spaces and Clopen Sets

Theorem (B.O. [2013])

For arbitrary strictly increasing sequence τ of positive integers the mapping f_w is an isometry from the space ΩT_τ of all clopen subsets of the boundary ∂T_τ equpped with the metric d_μ to the Hamming space of τ-periodic $(0, 1)$-sequences.

Group of Homeomorphisms

Now we consider the set $C(\partial T_\tau, S_2)$ of all continuous function from ∂T_τ to S_2. Define a binary operation $*$ on this set. For any $f, g \in C(\partial T_\tau, S_2)$

$$(f * g)(x) = f(x) \cdot g(x)$$

for all $x \in \partial T$. Then $C(\partial T, S_2)$ with operation $*$ is a group. Denote by $(Homeo\partial T_\tau \cap Aut(\partial T_\tau, \mu))$ the group of all homeomorphisms of the boundary ∂T_τ that preserve the measure μ.

Group of Homeomorphisms

The group $(Homeo\partial T_\tau \cap Aut(\partial T_\tau, \mu))$ acts on $C(\partial T_\tau, S_2)$ by generalized translations. Specifically, for $g \in (Homeo\partial T_\tau \cap Aut(\partial T_\tau, \mu))$ and $h \in C(\partial T_\tau, S_2)$ let

$$h^g(x) = h(x^g), x \in \partial T_\tau.$$

This action is an automorphism of $C(\partial T_\tau, S_2)$. Consequently, we can consider the semidirect product $C(\partial T_\tau, Z_2) \rtimes (Homeo\partial T_\tau \cap Aut(\partial T_\tau, \mu))$.

Periodic Hamming Spaces and Homeomorphisms

Theorem (B.O., V. Sushchansky [2013])

Let u be a supernatural number and let $\tau = (m_1, m_2, \ldots)$ *be a strictly increasing divisible sequence of positive integers with* $char(\tau) = u$. *The isometry group* $Isom\mathcal{H}(u)$ *of the u-periodic Hamming space* $\mathcal{H}(u)$ *is isomorphic as a transformation group to the semidirect product*

$$C(\partial T_\tau, Z_2) \rtimes (Homeo\partial T_\tau \cap Aut(\partial T_\tau, \mu)),$$

where T_τ *is the spherically homogeneous rooted tree and* μ *is the Bernoulli measure on the* σ*-algebra of clopen sets of* ∂T_τ.

Isometries of the Besicovitch-Hamming Space

Denote by $Fun_\mu(\partial T_\tau, S_2)$ the group of measurable functions from ∂T_τ to S_2.

Theorem (B.O., V. Sushchansky [2013])

The isometry group $Isom\mathcal{H}$ of the Besicovitch-Hamming space $\mathcal{H}$ is isomorphic as a transformation group to the semidirect product

$$Fun_\mu(\partial T_\tau, S_2) \rtimes Aut(\partial T_\tau, \mu),$$

where T_τ is the spherically homogeneous rooted tree and μ is the Bernoulli measure on the σ-algebra of clopen sets of ∂T_τ.

Problems of P. J. Cameron and S. Tarzi

What is the structure of the isometry group of the periodic Hamming space over some finite alphabet? What is the structure of the isometry group of its completion?

Hyperoctahedral Groups

The group W_n consists of all pairs $[\sigma, f]$, where $\sigma \in S_n$, $f \in Z_2^{\underline{n}}$, $\underline{n} = \{1, \ldots, n\}$. Denote $f(i) = a_i$, $(1 \leq i \leq n)$. Each pair $[\sigma, f]$ corresponds to a unique sequence $[\sigma; a_1, \ldots, a_n]$. Then the group operation in $Z_2 \wr S_n$ is determined by the equality

$$[\sigma; a_1, \ldots, a_n][\eta; b_1, \ldots, b_n] = [\sigma\eta; a_1 + b_{1^\sigma}, \ldots, a_n + b_{n^\sigma}],$$

where $+$ denotes the addition in Z_2.

Hyperoctahedral Groups

The inverse of the element $[\sigma; a_1, \dots, a_n]$ is the element

$$[\sigma^{-1}; a_{1^{\sigma^{-1}}}, \dots, a_{n^{\sigma^{-1}}}].$$

A transformation $u = [\sigma; a_1, \dots, a_n]$ acts on the vector $\bar{t} = (t_1, \dots, t_n) \in Z_2^n$ according to the rule

$$t^u = (t_{1^\sigma} + a_1, \dots, t_{n^\sigma} + a_n).$$

Direct Limits of Hyperoctahedral Groups

Define an embedding of the permutation group $(W_{m_i}, Z_2^{m_i})$ into the permutation group $(W_{m_{i+1}}, Z_2^{m_{i+1}})$ by a pair of maps

$$h_i : W_{m_i} \to W_{m_{i+1}}, \qquad \delta_i : Z_2^{m_i} \to Z_2^{m_{i+1}},$$

such that for each $i \in \mathbb{N}$ we have:

1. $h_i([\sigma; a_1, \ldots, a_{m_i}]) = [\theta^{s_{i+1}}\sigma; \underbrace{(a_1, \ldots, a_{m_i}, \ldots, a_1, \ldots, a_{m_i})}_{m_i \cdot s_{i+1}}],$
2. $\delta_i(t_1, \ldots, t_{m_i}) = \underbrace{(t_1, \ldots, t_{m_i}, t_1, \ldots, t_{m_i}, \ldots, t_1, \ldots, t_{m_i})}_{m_i \cdot s_{i+1}},$

where $\sigma \in S_{m_i}$, $(a_1, \ldots, a_{m_i}), (t_1, \ldots, t_{m_i}) \in Z_2^{m_i}$ and

$$\theta^{s_{i+1}}\sigma = \left(\begin{array}{ccc|c|ccc} 1 & \ldots & m_i & \ldots & (s_{i+1}-1)m_i + 1 & \ldots & s_{i+1}m_i \\ 1^\sigma & \ldots & m_i^\sigma & \ldots & (s_{i+1}-1)m_i + 1^\sigma & \ldots & (s_{i+1}-1)m_i + m_i^\sigma \end{array} \right)$$

Direct Limits of Hyperoctahedral Groups

The increasing divisible sequence $\tau = (m_1, m_2, \ldots)$ determines the direct spectrum

$$\langle (W_{m_i}, Z_2^{m_i}), F_i \rangle_{i \in \mathbb{N}}. \tag{2}$$

of hyperoctahedral groups $(W_{m_i}, Z_2^{m_i})$.
We call the direct limit of directed system (2) the *D-hyperoctahedral group* corresponding to the sequence τ and denote it by $W(\tau)$.

Isomorphic D-Hyperoctahedral Groups

Theorem (B.O., V. Sushchansky [2014])

Let τ_1, τ_2 be increasing divisible sequences. The groups $W(\tau_1)$ and $W(\tau_2)$ are isomorphic if and only if $char\tau_1 = char\tau_2$.

Metric Groups of Homeomorphisms

Equip the group of homeomorphisms $Homeo\partial T_\tau$ and the group $C(\partial T_\tau, Z_2)$ with the metrics

$$\sigma_\tau(f,g) = \max_{x \in \partial T_\tau} \rho_\tau(x^g, x^f), \qquad \text{for all } f,g \in Homeo\partial T_\tau,$$

$$\hat{\sigma}_\tau(h,t) = \begin{cases} 1, & \text{if } h \neq t \\ 0, & \text{if } h = t \end{cases}, \qquad \text{for all } h,t \in C(\partial T_\tau, Z_2).$$

Isometry Groups of Periodic Hamming Spaces

Theorem (B.O., V. Sushchansky [2013])

The isometry group $\mathcal{H}(u)$ of the u-periodic Hamming space $\mathcal{H}(u)$ is the closure of D-hyperoctahedral group $W(\tau)$, $char\tau = u$, regarded as a subgroup of $C(\partial T_\tau, Z_2) \rtimes Homeo\partial T_\tau$ in the Tychonoff product of topologies induced by the metrics σ_τ and $\hat{\sigma}_\tau$.

Normal Structure of D-Hyperoctahedral Groups

Let $B(u)$ be the subgroup of $W(u)$ consisting of elements of the form

$$[e, a_1, a_2, \ldots], \quad a_i \in Z_2.$$

We denote by $B_0(u)$ the subgroup of sequences $[e; a_1, a_2, \ldots] \in B(u)$ such that for certain number n, $n \mid u$, the equality

$$a_1 + a_2 + \ldots a_n = 0(mod 2)$$

holds.

Normal Structure of D-Hyperoctahedral Groups

Denote by C the subgroup of $B(u)$ containing only sequences $[e, 0, 0, \ldots]$ and $[e, 1, 1, \ldots]$. Define subgroups of the group $W(u)$ by the rule:

$$U = S(u) \cdot \mathcal{B}_0(u), \quad V = gp(W'(u), [(1,2); 1, 0, \ldots]), \quad H = A(r) \cdot \mathcal{B}(u),$$

where $gp(X)$ is the subgroup generated by the set X.

Normal Structure of D-Hyperoctahedral Groups

Theorem

Let u be an infinite supernatural number.
1) If $2^{\infty} \nmid u$, then the lattice of normal subgroups of the group $W(u)$ has the form depicted on Figure 2 in case $2 \mid u$ and the form depicted on Figure 3 in case $2 \nmid u$.
2) If $2^{\infty} \nmid u$ then the lattice of normal subgroups of the group $W(u)$ has the form depicted on Figure 4.

Normal Structure of D-Hyperoctahedral Groups

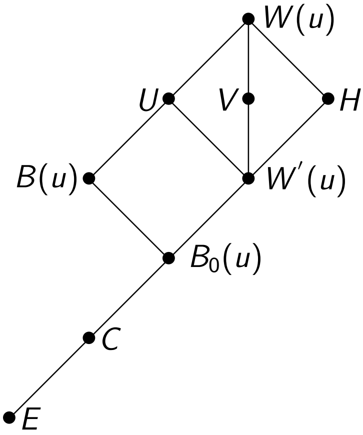

Figure 2:

Normal Structure of D-Hyperoctahedral Groups

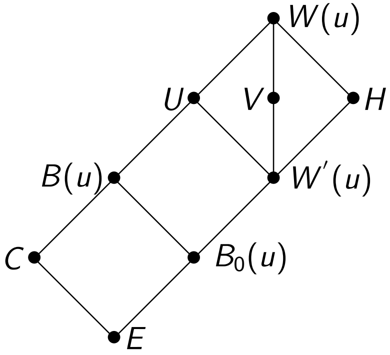

Figure 3:

Normal Structure of D-Hyperoctahedral Groups

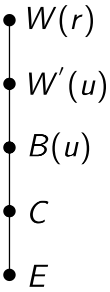

Figure 4:

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Thank you for your attention!