Critical groups isospectral to $U_3(3)$

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By a *section* of $G$ we mean a quotient group $H/N$, where $N, H \leq G$ and $N \leq H$. 
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Denote by $h(G)$ the number of pairwise non-isomorphic groups isospectral to $G$. We call $G$ \textit{recognizable} (by its spectrum) if $h(G) = 1$, \textit{almost recognizable} if $1 < h(G) < \infty$, and \textit{unrecognizable} if $h(G) = \infty$. 

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The *recognizability problem* for a group $G$ is said to be solved if $h(G)$ is known and if $1 < h(G) < \infty$ then also all groups isospectral to $G$ are described.
Let $\omega$ be a subset of natural numbers. A group $G$ is called \textit{critical with respect to} $\omega$ (or $\omega$-\textit{critical}) if $\omega$ coincides with the spectrum of $G$ and does not coincide with the spectra of proper sections of $G$.

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- If $h(G) = \infty$ then there exists a group $H$ isospectral to $G$ that contains a nontrivial soluble normal subgroup.
- For every set $\omega$ the number of $\omega$-critical groups is finite.
Conjecture (Mazurov V. D.)

For any natural number $n$ there exists a set $\omega$ such that there exist at least $n$ pairwise non-isomorphic groups critical with respect to $\omega$. 
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Question

Does there exist a natural number $n$ such that if $\omega$ is the spectrum of a non-abelian simple group then the number of pairwise non-isomorphic $\omega$-critical groups does not exceed $n$? If it exists then what is it?
Non-abelian simple groups:

- alternating;
- sporadic;
- classical;
- exceptional.

If $\omega$ is the spectrum of a non-abelian simple alternating or sporadic group then the number of pairwise non-isomorphic $\omega$-critical groups does not exceed 3.
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Lytkin Y. V. Groups critical with respect to the spectra of alternating and sporadic groups // Siberian Mathematical Journal. 2015. V. 56, N 1, P. 101–106.
If $\omega$ is the spectrum of an exceptional group then the number of pairwise non-isomorphic $\omega$-critical groups equals 1.

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The number of groups critical with respect to the spectrum of a special linear group $L_3(3)$ equals 2.

• Let $G$ be a group isospectral to a non-abelian simple group $S$. Then
  1. if $G$ is a Frobenius group then $S$ is isomorphic to $L_3(3)$ or $U_3(3)$;
  2. if $G$ is a double Frobenius group then $S$ is isomorphic to $U_3(3)$ or $S_4(3)$.

Aleeva M. R. On finite simple groups with the set of element orders as in a Frobenius group or a double Frobenius group // Mathematical Notes. 2003. V. 73, N 3-4. P. 299–313.
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There are constructed examples of a Frobenius group and a double Frobenius group isospectral to $U_3(3)$.
Theorem

Let $G$ be a group critical with respect to the spectrum of $U_3(3)$ that contains a normal subgroup $N$ such that $G/N \cong PGL_2(7)$. Then $N$ is elementary Abelian of order $2^6$ and there exists a subgroup $H \cong PGL_2(7)$ of $G$, such that $G = NH$.

Moreover, $H$ has a presentation

$$\langle a, b, c \mid a^2 = b^3 = c^2 = (ab)^7 = (ac)^2 = (bc)^2 = [a, b]^4 = 1 \rangle,$$

and if we regard $N$ as a vector space over $GF(2)$, then we can choose a base in $N$ in which the action of $H$ on $N$ is defined by the following matrices:

$$a \sim \begin{pmatrix} 1 & \cdots & \cdots & \cdots & 1 \\ 1 & \cdots & \cdots & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & 1 \\ 1 & \cdots & \cdots & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \end{pmatrix}, \quad b \sim \begin{pmatrix} \cdots & \cdots & \cdots & 1 & \cdots \\ 1 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \end{pmatrix}, \quad c \sim \begin{pmatrix} \cdots & \cdots & \cdots & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \end{pmatrix}.$$