On the (subgroup) isomorphism problem of group rings

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Groups and their actions

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$V(RG)$	group of normalized units of RG, i.e.

$$V(RG) = \left\{ \sum_{g \in G} u_g g \in \mathrm{U}(RG) : \sum_{g \in G} u_g = 1 \right\}$$

Fundamental Questions

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- Given the integral group ring $\mathbb{Z}G$. Which properties of G are determined by $\mathbb{Z}G$ (G.Higman 1940) ?
- Given the character table $X(G)$ of a finite group G. What can be said over the structure of G ?

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- Even if one assumes that the integral group rings $\mathbb{Z}G$ and $\mathbb{Z}H$ are isomorphic it does not follow in general that G and H are isomorphic (M.Hertweck 1998, Annals of Math.).

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- However for some classes of finite groups Brauer's question has an affirmative answer.
- The isomorphism problem for integral group rings is in some sense **almost** true, will be explained later.

Related Conjectures, Group Rings of Simple Groups

Theorem (K.-Nagl 2002, 2007)

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1. Let G be simple of Lie type with defining characteristic p and let K be a field of characteristic p, then $KG \cong KH$ implies that H is simple.

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- The case for the alternating groups in Step 2 has been completed by M.Nagl in his Diplomarbeit 2007.

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3 By step 1 and 2 we get that

$$KG \cong KH \ \ \forall K \implies H \text{ is simple and } |H| = |G|.$$

By Artin's theorem on the order of the finite simple groups it follows that $G \cong H$ except $\{G, H\} = \{A_8, A_2(4)\}$ or $= \{B_n(q), C_n(q)\}$.

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4. Clearly $\mathbb{C}A_8 \not\cong \mathbb{C}A_2(4)$.
5. $\mathbb{C}B_n(q) \not\cong \mathbb{C}C_n(q)$ because their smallest ordinary character degrees $\neq 1$ are different (Tiep and Zalesski).

Related Conjectures, Group Rings of Simple Groups III

The proof of the Theorem above suggests the following.

Modification of Brauer's Problem 2* for simple groups

Let G be a finite simple group. Assume that $\mathbb{C}G \cong \mathbb{C}H$. Are then G and H isomorphic ?

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Modification of Brauer's Problem 2* for simple groups

Let G be a finite simple group. Assume that $\mathbb{C}G \cong \mathbb{C}H$. Are then G and H isomorphic ?

Note that in the case of a positive answer it follows that the rational group algebra $\mathbb{Q}G$ determines a simple group up to isomorphism.

Results counting multiplicities II

Theorem (Hung Tong Viet 2010)

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Independently M.Nagl could show this 2009 but only for simple groups of Lie type over a prime field (with slightly different methods).

$B_n(q)$ versus $C_n(q)$

Proposition (T.Vassias 2011)

Let p be the defining characteristic of $B_n(q)$ with q odd. Let K be a field of characteristic p. Then

$$KB_n(q) \ncong KC_n(q).$$

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Corollary

For each finite simple group of Lie type there is a field K of positive characteristic such that KG determines G up to isomorphism.

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Proposition (jt. work with M.Nagl)

Let G be a finite simple group. Assume that G has a cyclic Sylow p - subgroup. Then F_pG determines G up to isomorphism.

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Proposition (jt. work with M.Nagl)

Let G be a finite simple group. Assume that G has a cyclic Sylow p - subgroup. Then $F_p G$ determines G up to isomorphism.

Note that each finite simple group has a cyclic Sylow subgroup for some prime p .

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Now Tong Viets Theorem yields the result.

Some basic facts on torsion subgroups of $V(\mathbb{Z}G)$

Let U be a torsion subgroup of $V(\mathbb{Z}G)$

- U is finite and $|U|$ divides $|G|$.
- If U is central then $U < G$.
- If $|U| = |G|$ then $\mathbb{Z}G = \mathbb{Z}U$. Such a U is called a **group basis**
- In general a torsion subgroup is **not contained** in a group basis.

Description of Integral Group Rings

A typical element of $\mathbb{Z}G$ may be written as
$\sum_{g \in G} z_g \cdot g, \ z_g \in \mathbb{Z}$.
One can also consider $\mathbb{Z}G$ as order in $\mathbb{Q}G$ or $\mathbb{C}G$

Example. Let $G = S_3 =< a, b; a^3, b^2, a^b = a^2 >$. Then

$$\mathbb{Z}G \subset \mathbb{Z} \times \mathbb{Z} \times \begin{pmatrix} \mathbb{Z} & 3\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$$

$x = a_1 \times a_2 \times \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is an element of $\mathbb{Z}G$ if and only if
$a_1 \equiv a_2 \ mod \ 2, a_1 \equiv a_{11} \ mod \ 3, a_2 \equiv a_{22} \ mod \ 3.$

$u = 1 \times -1 \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ corresponds to
$u = a - a^2 - b + ab + a^2 b$ in the other description.

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Hertwecks's counterexample (of order $2^{21} \cdot 97^{28}$) shows that in general the answer is no!
Not each finite subgroup of $V(\mathbb{Z}G)$ is isomorphic to a subgroup of G. But for many classes of groups the isomorphism problem has a positive answer.

Positive Results on IP

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abelian - by - nilpotent (Roggenkamp - Zimmermann)

G is semisimple (K.-Lyons-Sandling-Teague).

Positive Results II

F^* - Theorem (Roggenkamp - Scott)

Suppose that the generalized Fitting subgroup $F^*(G)$ is a p - group. Then $\mathbb{Z}G$ determines G up to isomorphism. Moreover group bases are rationally conjugate.

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Corollary

For each finite group G exists an abelian extension $E = A \cdot G$ such that $\mathbb{Z}E$ determines E up to isomorphism.

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Which finite subgroups of $V(\mathbb{Z}G)$ are isomorphic to subgroups of G?

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Classify all finite groups H with the following property.

Suppose that H occurs as subgroup of $V(\mathbb{Z}G)$, where G is finite. Then H is isomorphic to a subgroup of G.

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A result of Cohn - Livingstone 1963 gives a positive answer to the subgroup isomorphism problem for cyclic p -groups.

Till 2006 this result was the only general result known on the subgroup isomorphism problem.

Open questions I

In particular the following important cases of the subgroup isomorphism problem are open.

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Open questions I

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- **Question A** Is a finite abelian subgroup of $V(\mathbb{Z}G)$ always isomorphic to a subgroup of G?
- **Question C** In particular, are finite cyclic subgroups of $V(\mathbb{Z}G)$ isomorphic to a subgroups of G?
- **Question P** Is a finite p - subgroup of $V(\mathbb{Z}G)$ always isomorphic to a subgroup of G?

Open questions II

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If true this provides a strong affirmative answer to Question C.

It motivates in addition to SIP to ask which isomorphic torsion subgroups of $V(\mathbb{Z}G)$ are rationally conjugate.

Known special results

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We know quite a lot in the case when G is a soluble or nilpotent group, e.g.

- finite cyclic subgroups of $V(\mathbb{Z}G)$ are isomorphic to a subgroup of G provided G is finite soluble (Hertweck 2007).
- all torsion subgroups of $V(\mathbb{Z}G)$ are conjugate within $\mathbb{Q}G$ to a subgroup of G provided G is finite nilpotent (Weiss 1991).

First results on non-cyclic subgroups

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- Let p be an odd prime. $V(\mathbb{Z}G)$ contains subgroups isomorphic to $C_p \times C_p$ if, and only if, G has such a subgroup. (Hertweck 2007)

First results on non-cyclic subgroups

- The subgroup isomorphism problem has an affirmative answer if $H \cong C_2 \times C_2$. (K. 2006)
- Let p be an odd prime. $V(\mathbb{Z}G)$ contains subgroups isomorphic to $C_p \times C_p$ if, and only if, G has such a subgroup. (Hertweck 2007)
- Note that the proof for $p = 2$ uses the Brauer-Suzuki Theorem and is substantially different from that one for odd p.

Open Problems for ZG III

Question C contains as special case the following.

Prime graph question

Does the prime graph of $\mathbb{Z}G$ coincide with that one of G?

This should be seen as a first test for the Zassenhaus conjecture.

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This leads to the question whether Sylowlike theorems hold in $V(\mathbb{Z}G)$.

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Theorem (Abold - Plesken 1978)

In $\mathrm{GL}(\mathrm{n}, \mathbb{Z})$ p - subgroups are conjugate to a subgroup of a maximal p - subgroup. The conjugation takes place in $\mathrm{GL}(\mathrm{n}, \mathbb{Q})$.

Sylowlike Theorems in $\mathbb{Z}G$

Let G be a finite group.
We say that in $\mathbb{Z}G$ a **strong Sylow theorem** holds provided each p- subgroup of $V(\mathbb{Z}G)$ is conjugate within $\mathbb{Q}G$ to a p - subgroup of G

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Note that a p - subgroup of $V(\mathbb{Z}G)$ has order dividing $|G|$. Its exponent divides the exponent of G.

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and that a **weak Sylow theorem** holds if each p - subgroup of $V(\mathbb{Z}G)$ is isomorphic to a subgroup of G.

Note that a p - subgroup of $V(\mathbb{Z}G)$ has order dividing $|G|$. Its exponent divides the exponent of G.

It is an **open question** whether a strong Sylow theorem is valid in $\mathbb{Z}G$ for each finite group G.

Evidence for a positive answer, soluble groups

a) (Roggenkamp-Scott, Weiss 1987) $\mathbb{Z}G \cong \mathbb{Z}H$, G a p-group. Then a strong Sylow theorem holds in $V(\mathbb{Z}G)$.

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b) (Dokuchaev-Juriaans 1996) G nilpotent - by - nilpotent. Then in $\mathbb{Z}G$ a strong Sylow theorem holds.

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b) (Dokuchaev-Juriaans 1996) G nilpotent - by - nilpotent. Then in $\mathbb{Z}G$ a strong Sylow theorem holds.

c) (K. - Roggenkamp 1993) $\mathbb{Z}G = \mathbb{Z}H$, G soluble. Then each p - subgroup of H is conjugate to a p - subgroup of G within $\mathbb{Q}G$.
i.e. a strong Sylow theorem for group bases holds.

Evidence for a positive answer, outside soluble groups

d) (Dokuchaev-Juriaans-Milies 1997) Suppose that G is a Frobenius group. Then a strong Sylow theorem holds for odd primes.
It holds for all primes provided G does not map onto S_5.

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e) (A.Bächle - K. 2011) Suppose that G has abelian Sylow 2 - subgroups of order ≤ 8. Then each 2 - subgroup of $V(\mathbb{Z}G)$ is rationally conjugate to a subgroup of G.

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It holds for all primes provided G does not map onto S_5.

f) (L.Margolis 2014) Let $G = \mathrm{PSL}(2, \mathrm{q})$ with $q = 2^f$ or $q = p^f$ with $f = 1$. Then in $\mathbb{Z}G$ a strong Sylow theorem holds.

e) (A.Bächle - K. 2011) Suppose that G has abelian Sylow 2 - subgroups of order ≤ 8. Then each 2 - subgroup of $V(\mathbb{Z}G)$ is rationally conjugate to a subgroup of G.

Weak Results

Theorem (A.Bächle - L.Margolis 2015

Let $G = PSL(2, p^3)$. Then in $V(\mathbb{Z}G)$ a weak Sylow theorem holds.

Weak Results ctd.

Proposition (K.2014)

Suppose that G has abelian Sylow 2 - subgroups then 2 - subgroups of $V(\mathbb{Z}G)$ are isomorphic to subgroups of G.

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Suppose that G has abelian Sylow 2 - subgroups then 2 - subgroups of $V(\mathbb{Z}G)$ are isomorphic to subgroups of G.

Corollary

Suppose that all Sylow subgroups of odd order are cyclic or of order p^2 and Sylow 2 - subgroups are abelian or generalized quaterniojn. Then a weak Sylow theorem holds in $V(\mathbb{Z}G)$.

Towards a Sylow theorem in the soluble case

In order to establish a Sylow theorem in the soluble case I believe that the following would be an important step.

.

Let N be a normal p - subgroup of the soluble group G. Suppose further that $O_{p'}(G) = 1$. The reduction $G \longrightarrow G/N$ induces a map

$$\tau : V(\mathbb{Z}G) \longrightarrow V(\mathbb{Z}G/N).$$

If U is torsion subgroup in $Ker\tau$, then U is conjugate in $\mathbb{Z}_p(G)$ to a subgroup of N.

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If U is torsion subgroup in $Ker\tau$, then U is conjugate in $\mathbb{Z}_p(G)$ to a subgroup of N.

A result of Hertweck shows that U is rationally (i.e. in $\mathbb{Q}G$) conjugate to a subgroup of $Ker\tau$. However in many situations this is too weak.

If U is an arbitrary p - subgroup of $V(\mathbb{Z}G)$, G as above, then it is in general not true that U is conjugate in $\mathbb{Z}_p(G)$ to a subgroup of G.

Towards a Sylow theorem in the general case

Let N be a minimal normal subgroup of G which is perfect.
Let τ be the induced map of $G \longrightarrow G/N$ on units

$$\tau : V(\mathbb{Z}G) \longrightarrow V(\mathbb{Z}G/N).$$

If U is torsion subgroup in $Ker\tau$, is U is conjugate in $Z_p(G)$ to a subgroup of N?

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Is there a good substitute in integral group rings ?

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Theorem (jt. work with I.Köster)

Let G be a finite soluble group. Then for each prime p the integral group ring $\mathbb{Z}G$ determines the size of the normalizer of a Sylow p - subgroup of G.

Thank you for your attention

Finally we investigate the question whether G and H have the same Sylow numbers provided $\mathbb{Z}G$ and $\mathbb{Z}H$ are isomorphic as rings. In other words the question is whether the order of the normalizer of a Sylow p - subgroup of G is determined by $\mathbb{Z}G$.

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For each prime p we denote by $n_p(G)$ the number of Sylow p - subgroups of G and $sn(G)$ denotes the set of all Sylow numbers of G for primes dividing $|G|$.

Finally we investigate the question whether G and H have the same Sylow numbers provided $\mathbb{Z}G$ and $\mathbb{Z}H$ are isomorphic as rings. In other words the question is whether the order of the normalizer of a Sylow p - subgroup of G is determined by $\mathbb{Z}G$.

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The weaker question whether $sn(G)$ is determined by $X(G)$ has been raised recently by Moreto.

Moreto's Results

Theorem (Moreto 2013)

If $sn(G) = \{a, b\}$ then G is the product of two nilpotent Hall subgroups.

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Because $X(G)$ determines whether G has nilpotent Hall π-subgroups (by K. - Sandling) this gives information on prime divisors of Sylow numbers. Moreto's result rescues the theorem of F.Luca that G is soluble provided $sn(G) = \{a, b\}$.

jt. work with I.Köster

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Proposition 2. Suppose that G is soluble. Then $\mathbb{Z}G$ determines the Sylow numbers of G.
The main ingredient for the proof of Theorem 2 is the F^* - theorem.

Related questions, character degrees

Let $X_1(G)$ = set of the degrees of the ordinary irreducible characters of G.

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Huppert's Conjecture (2000):

Let G and H be finite groups. Assume that G is simple and that $X_1(G) = X_1(H)$.
Then $H \cong G \times A$, where A is abelian.

Results on Character Degrees

- B.Huppert established his conjecture for all minimal simple groups, some other simple groups of Lie type of small Lie rank, A_7, A_8, A_9, A_{10}, for 18 of the sporadic simple groups. The remaining sporadic groups were verified by Alavi, Daneshkah, Wakefield and Tong-Viet 2011.

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- Several series of simple groups of Lie types of small Lie rank (Wakefield and Tong-Viet 2008 -2012)
- It seems somehow to be artificial not to count (additionally) the multiplicities of the degrees ?

Character Degrees with Multiplicities

Denote by $X_1^m(G)$ the list of the ordinary character degrees of G with multiplicities.

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Then

$X_1^m(G) = X_1^m(H)$ if and only if $\mathbb{C}G \cong \mathbb{C}H$.

Therefore the modified Brauer problem has an affirmative answer provided Hupperts conjecture holds.

Results counting multiplicities I

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Theorem (Berkovich, Chillag, Herzog), Proc. AMS 1992

The nonabelian finite groups with nonlinear characters of distinct degree are the following.

a) Extraspecial 2 - groups.
b) Frobenius groups of order $p^n(p^n - 1)$ for some prime power order p^n with abelian kernel of order p^n and a cyclic complement.
c) The Frobenius group of order 72 with complement isomorphic to Q_8.

2-subgroups, elementary abelian subgroups

Theorem A1 (M.Hertweck - C.Höfert - K. 2008)

Let $G = PSL(2, p^f)$. Then every finite 2-subgroup of $V(\mathbb{Z}G)$ is isomorphic to a subgroup of G. Each elementary abelian subgroup of $V(\mathbb{Z}G)$ is isomorphic to a subgroup of G.

Proposition A2 (A.Bächle - K. 2008)

For any prime p the elementary abelian p-subgroups of $V(\mathbb{Z}\mathrm{Sz}(\mathrm{q}))$ are isomorphic to subgroups of $\mathrm{Sz}(\mathrm{q})$.
If $p \in \{2, 5\}$, then this isomorphism can be taken as conjugation with a unit of $\mathbb{Q}\mathrm{Sz}(\mathrm{q})$.

Conclusion

Thank you for your attention