Diagonal direct limits of monomial groups

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Monomial groups
Finitary monomial group
Diagonal limits
Wreath product repr.
Conjugation in $Mon(G, t)$

Monomial groups

Recall some well known definitions and facts about classical monomial groups [W. Specht (1933), W.K. Turkin (1935), O.Ore (1942)].

Let G be a group, $n \in \mathbf{N}$, and let $x_1, x_2, ..., x_n$ be variables.
A monomial permutation (substitution) over G of variables $x_1, x_2, ..., x_n$ is a transformation of the type

$$u = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ g_1 x_{i_1} & g_2 x_{i_2} & \dots & g_n x_{i_n} \end{pmatrix},$$

where $\begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix} \in S_n$ and $g_1, g_2, ..., g_n \in G$.

Monomial groups
Finitary monomial group
Diagonal limits
Wreath product repr.
Conjugation in $Mon(G, t)$

Monomial groups

If $v = \left(\begin{array}{cccc} x_1 & x_2 & \dots & x_n \\ h_1 x_{j_1} & h_2 x_{j_2} & \dots & h_n x_{j_n} \end{array} \right)$ then

$$uv = \left(\begin{array}{cccc} x_1 & x_2 & \dots & x_n \\ g_1 h_{j_1} x_{i_{j_1}} & g_2 h_{j_2} x_{i_{j_2}} & \dots & g_n h_{j_n} x_{i_{j_n}} \end{array} \right) \quad (1)$$

The inverse of u is

$$u^{-1} = \left(\begin{array}{cccc} x_{j_1} & x_{j_2} & \dots & x_{j_n} \\ g_1^{-1} x_1 & g_2^{-1} x_2 & \dots & g_n^{-1} x_n \end{array} \right).$$

- The set $Mon(G, n)$ of all monomial permutations over G of variables $x_1, x_2, ..., x_n$ with multiplication defined in (1) forms a group which is called the complete monomial group of degree n over G.

Monomial groups
Finitary monomial group
Diagonal limits
Wreath product repr.
Conjugation in $Mon(G, t)$

Monomial groups

The group $Mon(G, n)$ contains two subgroups:

1. The subgroup of transformations of the form

$$\begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ ex_{i_1} & ex_{i_2} & \ldots & ex_{i_n} \end{pmatrix},$$

   which is isomorphic to S_n

2. The subgroup $D(G, n)$ of translations

$$\begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ g_1x_1 & g_2x_2 & \ldots & g_nx_n \end{pmatrix}, \quad g_1, g_2, ..., g_n \in G,$$

   which is isomorphic to $G \times G \times ... \times G$ (with n factors).

Monomial groups
Finitary monomial group
Diagonal limits
Wreath product repr.
Conjugation in $Mon(G, t)$

Monomial groups

Thus we obtain

$$Mon(G, n) \cong G \wr S_n. \tag{2}$$

The wreath product (2) is nowadays the most commonly used definition for the group of monomial permutations.

Yet another definition arises from the monomial matrix representation:

$$\begin{pmatrix} x_1 & x_2 & \dots & x_n \\ g_1 x_{i_1} & g_2 x_{i_2} & \dots & g_n x_{i_n} \end{pmatrix} \longrightarrow \begin{pmatrix} & & \dots & \\ g_1 & & \dots & \\ & & \dots & \\ & & \dots & g_n \\ & g_2 & \dots & \\ & & \dots & \end{pmatrix} \begin{matrix} \\ -i_1 \\ \\ -i_n \\ -i_2 \\ \\ \end{matrix}$$

$Mon(G, n)$ is isomorphic to the subgroup of monomial matrices in $GL(n, ZG)$.

Monomial groups
Finitary monomial group
Diagonal limits
Wreath product repr.
Conjugation in $Mon(G, t)$

Monomial groups

The two particular constructions of $Mon(G, n)$ are often used:

1. $G = C_k$ - the group of roots from 1 of degree k. The group $Mon(C_k, n) \cong C_k \wr S_n$ is often called the Frobenius monomial group.

2. $k = 2$. In this case the monomial permutations can be represented as

$$\begin{pmatrix} x_1 & x_2 & \dots & x_n \\ \pm x_1 & \pm x_2 & \dots & \pm x_n \end{pmatrix}.$$

   The group $Mon(C_2, n) \cong C_2 \wr S_n$ is called the group of signed permutations or the hyperoctahedral group (it is the group of symmetries of the n-dimensional cube).

Monomial groups
Finitary monomial group
Diagonal limits
Wreath product repr.
Conjugation in $Mon(G, t)$

Finitary monomial group

A finitary monomial matrix over group G is an infinitely dimensional matrix of the type

$$\left(\begin{array}{ccc} A & & 0 \\ & {\scriptstyle 1} & \\ 0 & & {\scriptstyle 1} \\ & & & \ddots \end{array}\right),$$

where $A \in Mon(G, n)$ for certain $n \in \mathbf{N}$.
All finitary monomial matrices over G form a group which is called the finitary monomial group and denoted by $Mon(G, \aleph_0)$. The group $Mon(G, \aleph_0)$ is the direct limit of the sequence $\langle Mon(G, n)\rangle, n \in \mathbf{N}$, with the natural embedding $Mon(G, n) \longrightarrow Mon(G, n+1)$

$$A \longrightarrow \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix},$$

Monomial groups
Finitary monomial group
Diagonal limits
Wreath product repr.
Conjugation in $Mon(G, t)$

Finitary monomial group

In terms of wreath product the finitary monomial group may be characterized as follows:

$$Mon(G, \aleph_0) \cong G \bar{\wr} FS_\infty,$$

where $\bar{\wr}$ denotes the bounded wreath product and FS_∞ is the finitary symmetric group of a countable set.
The finitary monomial group is the minimal infinite generalization of the complete monomial permutation groups. Other ones have been investigated by R. Crouch (1955), C.V. Holmes (1960,1961) and others.

Diagonal limits of monomial groups

Monomial groups
Finitary monomial group
Diagonal limits
Wreath product repr.
Conjugation in $Mon(G, t)$

In the theory of C^*-algebras a well known embedding of matrix algebras is the so called the diagonal embedding.

Let $k, l \in \mathbf{N}$ and $l = m \cdot k$.
The embedding $d_m : M_k(R) \longrightarrow M_l(R)$ defined as

$$d_m(A) = \begin{pmatrix} A & & & 0 \\ & A & & \\ & & \ddots & \\ 0 & & & A \end{pmatrix}$$

is called (strictly) diagonal embedding.

Diagonal limits of monomial groups

Let $\bar{k} = \langle k_1, k_2, \ldots \rangle$ be an infinite divisible sequence of positive integers, i.e.

$$k_{i+1} = m_i \cdot k_i, \quad m_i \in \mathbf{N} \quad for \quad i = 1, 2, \ldots .$$

We define the $\bar{k}$ - direct spectrum of monomial groups over G as $\langle Mon(G, k_i), d_{m_i} \rangle$.

Let $Mon(G, \bar{k}) = \varinjlim (Mon(G, k_i), d_{m_i})$.

The group $Mon(G, \bar{k})$ is called the $\bar{k}$ - diagonal limit monomial group over G.

Question: Classify $Mon(G, \bar{k})$, $\bar{k}$ - divisible sequences, up to isomorphism.

Diagonal limits of monomial groups

A supernatural number is a formal product $\prod_i p_i^{\alpha_i}$, where $p_1, p_2, \dots$ is the sequence of all primes and $\alpha_i \in \mathbf{N} \cup \{\infty\}$.

- Divisibility: $\prod_i p_i^{\alpha_i} | \prod_i p_i^{\beta_i} \quad \Leftrightarrow \quad \alpha_i \leq \beta_i, \quad n < \infty.$
- The partially ordered set SN of all supernatural numbers is a lattice. For $u = \prod_i p_i^{\alpha_i}$ and $v = \prod_i p_i^{\beta_i}$ we have:

$$u \vee v = \prod_i p_i^{max(\alpha_i, \beta_i)}$$

$$u \wedge v = \prod_i p_i^{min(\alpha_i, \beta_i)}$$

Monomial groups
Finitary monomial group
Diagonal limits
Wreath product repr.
Conjugation in $Mon(G, t)$

Monomial groups
Finitary monomial group
Diagonal limits
Wreath product repr.
Conjugation in $Mon(G, t)$

Diagonal limits of monomial groups

- The lattice SN of supernatural numbers is a complete distributive lattice.
- Every divisible sequence $\bar{k} = \langle k_1, k_2, \ldots \rangle$ defines a unique supernatural number

$$ch(\bar{k}) = k_1 \cdot \left(\frac{k_2}{k_1}\right) \cdot \left(\frac{k_3}{k_2}\right) \cdot \ldots$$

Examples:

- if $\bar{k} = \langle 1, 2, 2^2, 2^3, \ldots \rangle$ then $ch(\bar{k}) = 2^\infty$;
- if $\bar{k} = \langle 2, 6, 12, 36, 72, \ldots \rangle$ then $ch(\bar{k}) = 2^\infty \cdot 3^\infty$;
- if $\bar{k} = \langle 2, 4, 24, 72, \ldots \rangle$ then $ch(\bar{k}) = 2^2 \cdot 3^\infty$.

Diagonal limits of monomial groups

Monomial groups
Finitary monomial group
Diagonal limits
Wreath product repr.
Conjugation in $Mon(G, t)$

Let D be the set of all divisible sequences. The answer to the classification question is standard:

Th 1 *For every divisible sequences $\bar{k}, \bar{k}' \in D$ and a group G, the monomial groups $Mon(G, \bar{k})$ and $Mon(G, \bar{k}')$ are isomorphic iff $ch(\bar{k})$ is equal to $ch(\bar{k}')$:*

$$Mon(G, \bar{k}) \cong Mon(G, \bar{k}') \quad \Leftrightarrow \quad ch(\bar{k}) = ch(\bar{k}').$$

Thus we use the notation: $Mon(G, n)$, $n \in SN$.

This way we construct a continual family of pairwise non-isomorphic infinitely dimensional d-monomial groups.

Wreath product representation

We define the diagonal embedding d_r of the symmetric group S_n into the symmetric group S_{nr} as follows.

Permutation $d_r\alpha$, $\alpha \in S_n$, acts on the set $\{1, 2, ..., nr\}$ according to the rule:

$$(mn+i)^{d_r\alpha} = mn + i^{\alpha}, \quad 0 \le m \le r-1,\ 1 \le i \le n.$$

For every divisible sequence $\bar{k} = \langle k_1, k_2, ...\rangle$, $k_{i+1} : k_i = r_i$, we define the direct spectrum $\langle S_{k_i}, d_{r_i}\rangle_{i\in\mathbf{N}}$.

The limiting group of this spectrum is called a $\bar{k}$- homogeneous symmetric group. Similarly a $\bar{k}$-homogeneous alternating group can be defined. We denote these limits as $S(\bar{k})$ and $A(\bar{k})$.

Wreath product representation

Since

$$S(\bar{k}) \cong S(\bar{k}') \;\; (A(\bar{k}) \cong A(\bar{k}')) \quad \Leftrightarrow \quad ch(\bar{k}) = ch(\bar{k}')$$

then we may use the notation: $S(n)$, $A(n)$, for $n \in SN$.

The groups $S(n)$ and $A(n)$ for an arbitrary $n \in SN$ can be interpreted in a natural way as subgroups of S_∞.

Wreath product representation

Th 2

1. *The mapping* $t \mapsto S(t)$ ($t \mapsto A(t)$), $t \in SN$, *is an embedding of the lattice of supernatural numbers into the lattice of subgroups in* S_∞;
2. $S(t) = A(t)$ *iff* $2^\infty | t$;
3. *If* $2^\infty \nmid t$ *then* $[S(t) : A(t)] = 2$;
4. $A(t)$ *is a simple group.*

Monomial groups
Finitary monomial group
Diagonal limits
Wreath product repr.
Conjugation in $Mon(G, t)$

Wreath product representation

Let G be a group, $t \in SN$. An infinite sequence $g_1, g_2, \ldots$, $g_i \in G$ is called periodic if there exist number m such that $g_{i+ml} = g_i$ for $i = 1, 2, \ldots, m$, $l = 1, 2, \ldots$.

We say that an element $[\Pi; g_1, g_2, \ldots]$ of the wreath product $G \wr S(t)$ is t -periodically defined if the minimal period of the sequence $g_1, g_2, \ldots$ divides t.
All t-periodically defined elements of $G \wr S(t)$ form a subgroup, which we call the t - wreath product of groups $S(t)$ and G and denote $G \wr_t S(t)$.

Th 3 *For every supernatural number $t \in SN$ we have*

$$Mon(G, t) \cong G \wr_t S(t).$$

Conjugation in $Mon(G, t)$

Let t be a fixed supernatural number.

For every permutation $\alpha \in S(t)$ there exist a minimal number k and a permutation $\alpha_0 \in S(k)$ such that $\alpha = \lim_r d^r(\alpha_0)$.

If $(l_1, l_2, \ldots, l_k)$ is the cyclic type of α_0, then the vector $st(\alpha) = (l_1, \ldots, l_q)$ such that $l_q \neq 0$ and $l_{q+1} = \ldots = l_k = 0$ is called the short cyclic type of α.

Vectors $\bar{l} = (l_1, \ldots, l_q)$ and $\bar{l}' = (l'_1, \ldots, l'_q)$ are called t-similar if there exists m such that $m|t$ and either $m\bar{l} = \bar{l}'$ or $\bar{l} = m\bar{l}'$.

Conjugation in $Mon(G, t)$

Th 4 *The monomial transformations*

$$[\sigma; g_1, g_2, ...], [\tau; h_1, h_2, ...] \in Mon(G, t)$$

are conjugated iff:

1. $st(\sigma)$ *is* t*-similar to* $st(\tau)$*;*
2. *All products of the type* $\prod_{i=l}^{m} g_i$ *and* $\prod_{i=l}^{m} g_i$ *constructed for the respective cycles of permutations* σ *and* τ *are conjugated in* G*.*

Monomial groups
Finitary monomial group
Diagonal limits
Wreath product repr.
Conjugation in $Mon(G, t)$

Thank you.