Element orders in coverings of finite simple classical groups

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Recognition by spectrum and coverings

A group $H$ is a covering of a group $G$ if $G = H/K$ for some normal subgroup $K$ of $H$. If $H$ is a covering of $G$ then $\omega(G) \subseteq \omega(H)$. We say that $G$ is recognizable by spectrum among coverings if $\omega(G) \subset \omega(H)$ for every proper covering $H$ of $G$. 

Lemma 1

A group $G$ is recognizable by spectrum among coverings if and only if $\omega(G) \subset \omega(H)$ for every split extension $H = K \rtimes G$, where $K$ is an elementary abelian group and $G$ acts on $K$ absolutely irreducibly.

Lemma 2

If $K$ is a normal abelian subgroup of a group $H$ then $\omega(H) = \omega(K \rtimes H)$, where $H$ acts on $K$ via conjugation.
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If $G$ is not recognizable by spectrum among coverings then
\[ \omega(G) = \omega(K \rtimes G) \] for some abelian $K \neq 1$ and
\[ \omega(G) = \omega(K \rtimes G) = \omega(K \rtimes (K \rtimes G)) = \ldots \]

and thus there are infinitely many finite groups isospectral to $G$, i.e., $h(G) = \infty$.

On the other hand, if a finite simple group $G$ is quasirecognizable and recognizable among coverings then $h(G) < \infty$.

It is not true that $h(G) = \infty$ implies that $G$ is not recognizable among coverings. Recently Mazurov proved that $h(G) = \infty$ implies that $\omega(G) = \omega(H)$ for some finite group $H$ having a nontrivial normal abelian subgroup.
Let $L$ be a finite simple classical group over a field of characteristic $p$. Suppose $L$ acts on an elementary abelian $r$-group some prime $r$. It is natural to distinguish the case $r = p$ from the case $r \neq p$.

$(C_p)$ $\omega(K \rtimes L) \neq \omega(L)$ for every elementary abelian $p$-group $K$

$(C_{p'})$ $\omega(K \rtimes L) \neq \omega(L)$ for every elementary abelian $p'$-group $K$
Previous results

The problem was completely solved for linear groups and partially for unitary groups by A. Zavarnitsine and V. Mazurov.

<table>
<thead>
<tr>
<th>Simple group</th>
<th>$C_p$</th>
<th>$C_{p'}$</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_n(q)$, $n \geq 4$</td>
<td>+ if $n \neq 4$</td>
<td>+</td>
<td>Zav08, ZavMaz07, Zav00</td>
</tr>
<tr>
<td>$U_n(q)$, $n \geq 4$</td>
<td>+ if $n \neq 4$</td>
<td></td>
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<tr>
<td>$S_n(q)$, $n \geq 6$</td>
<td></td>
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<tr>
<td>$O_n^\varepsilon(q)$, $n \geq 7$</td>
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</tbody>
</table>

The table does not contain groups of small dimensions since the recognition-by-spectrum problem is solved for them. The sign ”+“ means that groups have the corresponding property. The question about $C_p$ for $L_4(q)$ and $U_4(q)$ is still open.
# Results

<table>
<thead>
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<tr>
<td>$L_n(q), \ n \geq 4$</td>
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<tr>
<td>$U_n(q), \ n \geq 4$</td>
<td></td>
<td>+ if $n \neq 5$</td>
<td>Gr10</td>
</tr>
<tr>
<td>$S_n(q), \ n \geq 6$</td>
<td></td>
<td>+</td>
<td>Gr10</td>
</tr>
<tr>
<td>$O_n^e(q), \ n \geq 7$</td>
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<td>+</td>
<td>Gr10</td>
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Theorem 1
Let $L$ be one of the groups $S_n(q)$, where $n \geq 6$, and $O^\varepsilon_n(q)$, where $n \geq 7$, $r$ be a prime and $(q, r) = 1$. If $L$ acts on an elementary abelian $r$-group $K$ then $\omega(K \rtimes L) \neq \omega(L)$.

Theorem 2
Let $L = U_n(q)$, where $n \geq 4$, $r$ be a prime and $(q, r) = 1$. If $L$ acts on an elementary abelian $r$-group $K$ then either $\omega(K \rtimes L) \neq \omega(L)$ or one of the following holds:
(i) $L = U_5(p)$, where $p$ is a Mersenne prime and $r = 2$;
(ii) $L = U_5(2)$, $q = 2$ and $r = 3$. 
Methods: Frobenius subgroups

Lemma 3
If a Frobenius group $G$ with kernel $F$ and cyclic complement $C$ acts faithfully on an elementary abelian $r$-group $K$ and $|F|$ is not divisible by $r$ then $r|C| \in \omega(K \rtimes G)$. 

Lemma 3

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The idea is to find a Frobenius subgroup with “large” complement in $L$. This trick works excellent for linear and symplectic groups and imposes a powerful restriction on $K$ for other groups except unitary groups of small dimensions.
Theorem (DiMartino, Zalesskii)

Let $G$ be a finite classical group in characteristic $p$. Let $s \neq p$ be a prime and $g \in G$ be a non-central element such that $g$ belongs to a proper parabolic subgroup of $G$ and $|g|$ is a power of $s$. Let $\Phi$ be an absolutely irreducible representation of $G$ of degree $> 1$ over a field of characteristic $r \neq p$. Then either $d_\Phi(g) = |g|$ or $G$ is a symplectic or unitary group.

Moreover, Guralnick, Magaard, Saxl and Tiep showed that $d_\Phi(g) < |g|$ implies that $\Phi$ is a Weil representation.
**Theorem (DiMartino, Zalesskii)**

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Moreover, Guralnick, Magaard, Saxl and Tiep showed that $d_\Phi(g) < |g|$ implies that $\Phi$ is a Weil representation.

The idea is to find an element $g$ in a proper parabolic subgroup of $L$ such that $|g|$ is the largest $r$-power in $\omega(L)$. Then either $r|g| \in \omega(K \rtimes L) \setminus \omega(L)$ or an exception from the above theorem arises, in particular $L$ is symplectic or unitary.
To complete the investigation of recognizability among coverings for classical simple groups, it remains to answer the following question.

**Problem**

Suppose that a finite simple symplectic or orthogonal group $L$ with defining characteristic $p$ acts on an elementary abelian $p$-group $K$. Is it true that $\omega(K \rtimes L) \neq \omega(L)$?