On The Width Of Verbal Subgroups In Groups Of Unitriangular Matrices

Agnieszka Bier

Silesian University of Technology, Gliwice

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Outline

Notation

Words

Let X_∞^* denote a free group of countably infinite rank, freely generated by $X = \{x_1, x_2, ...\}$.

Elements of X_∞^* are called words. Every word $w \in X_\infty^*$ can be written in the form:

$$w(x_1, x_2, ..., x_k) = x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} ... x_{i_s}^{\epsilon_s},$$

where $x_{i_j} \in \{x_1, x_2, ..., x_k\} \subseteq X$ and $\epsilon_j \in \mathbb{Z} \setminus \{0\}$ for $j = 1, 2, ..., s$

Word values

Def 1 *An element $g \in G$ such that*

$$g = v(\underline{g}) = v(g_1, g_2, ..., g_n), \qquad \underline{g} = (g_1, g_2, ..., g_n) \in G^n$$

is called the value of the word v in group G

- $Val(v, G)$ - the set of all values of the word v in group G.

Def 2 *Let $W = \{w_i\}_{i \in I} \subseteq X_\infty^*$ be a set of words and let G be a group. The subgroup $V_W(G)$ generated by the set*

$$Val(W, G) = \bigcup_{w_i,\ i \in I} Val(w_i, G)$$

is called the verbal subgroup of G generated by W.

- If $W = \{w\}$, we simply write $V_w(G)$ instead of $V_W(G)$.

Verbal subgroups - examples

1. The words $c_1 = x_1, \quad c_{i+1} = [c_i(x_1, ..., x_i), x_{i+1}]$ for $i = 1, 2, ...$ are called the left-normed basic commutators.

For any group G the verbal subgroups $V_{c_i}(G)$ constitute the lower central series $G = \gamma_1(G) > \gamma_2(G) > ...$, in which

$$\gamma_i(G) = V_{c_i}(G).$$

Def 3 *If $\gamma_{i+1}(G) = \{e\}$ for some $i \in \mathbb{N}$ then G is called nilpotent of class i.*

Verbal subgroups - examples

2. The commutator words

$$\begin{aligned} d_1(x_1) &= x_1, \\ d_{i+1}(x_1, x_2, ..., x_{2^i}) &= [d_i(x_1, x_2, ..., x_{2^{i-1}}), d_i(x_{2^{i-1}+1}, ..., x_{2^i})] \end{aligned}$$

for $i = 1, 2, ...$ generate in every group G the verbal subgroups $V_{d_i}(G)$, which constitute the derived series of this group:

$$G \geq G' \geq G'' \geq G^{(3)} \geq ...,$$

where $G^{(i)} = V_{d_{i+1}}(G)$.

In particular, the derived subgroup G' of group G is its verbal subgroup $V_{d_2}(G)$.

Properties of verbal subgroups

- invariant to endomorphisms of the group (fully invariant subgroup)
- all verbal subgroups in a group constitute the lattice
- homomorphic image of verbal subgroup is a verbal subgroup in the homomorphic image of the group

Width of verbal subgroups

Let $W = \{w_i\}_{i \in I} \subseteq X_\infty^*$ be a set of words. Every element $v \in V_W(G)$ can be represented as a product of values of words w_i, $i \in I$.

Def 4 *The smallest number n, such that every element $v \in V_W(G)$ can be represented as a product of n values of some of the words w_i in G, $i \in I$ is called the width of the verbal subgroup and denoted by $wid_W(G)$.*
If no such a number exists, then $wid_W(G) = \infty$.

- If $V_W(G) = Val(W, G)$, then $wid_W(G) = 1$.
- Verbal width depends on the set of generating words.

Width of verbal subgroups - examples

1. Let A be an abelian group. For an arbitrary word $w \in X_\infty^*$
$$wid_w(A) = 1.$$
2. Let S_3 denote the symmetric group on a set of three elements and A_3 denote the respective alternating group. Then $V_{c_2}(S_3) = S_3' = A_3$ and
$$wid_{c_2}(S_3) = 1$$

Width of verbal subgroups - examples

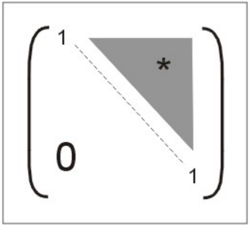
3. Allambergenov, Romankov, Smirnova:
   - $wid_{c_2}(N_{n2}) = \left[\frac{n}{2}\right]$ for $n > 1$
   - $wid_{c_2}(N_{nk}) = n$ for $k \geq 3$
   - $wid_{c_2}(M_n) = n$
   - $wid_{x^{2k}}(N_{n2}) = 2\left[\frac{n}{2}\right] + 1$ for $n > 1$, $k > 0$
   - $wid_{x^{2k+1}}(N_{n2}) = 1$

N_{nk} - free nilpotent group of rank n and class k,
M_n - free metabelian group of rank n

Groups of unitriangular matrices

Groups of unitriangular matrices

- K - an arbitrary field
- $UT_n(K)$ - the group of unitriangular $n \times n$ matrices over field K



Groups of unitriangular matrices

- $\mathbf{1}_n \in UT_n(K)$ - identity matrix
- e_{ij} - $n \times n$ matrix with 1 in the place (i,j) and zeros elsewhere
- Every matrix $A \in UT_n(K)$ can be written as:

$$A = \mathbf{1}_n + \sum_{1 \leq i < j \leq n} a_{ij} e_{ij}, \qquad a_{ij} \in K.$$

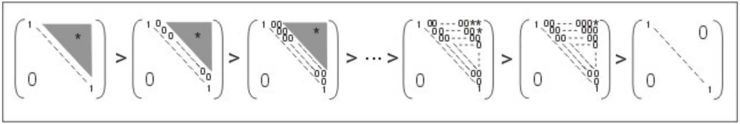
Groups of unitriangular matrices

- The lower central series of $UT_n(K)$:

$$UT_n(K) = UT_n^0(K) > UT_n^1(K) > \ldots > UT_n^{n-1}(K) = \{\mathbf{1_n}\},$$

where

$$UT_n^l(K) = \left\{ \mathbf{1}_n + \sum_{i<j-l\leq n} a_{i,j} e_{i,j}, \;\; a_{i,j} \in K \right\}, \;\; 0 \leq l \leq n-1.$$

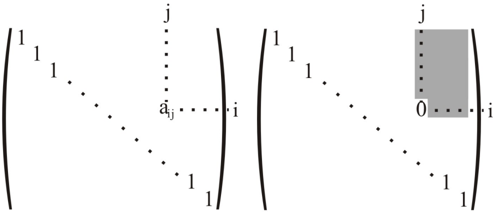


Characteristic subgroups - Weir, Levchuk

We denote:

$$P_{ij} = \{\mathbf{1_n} + ae_{ij} \mid a \in K\},$$

$$Q_{ij} = P_{1,j}P_{1,j+1}\dots P_{1,n}P_{2,j}\dots P_{2,n}\dots P_{i-1,n}P_{i,j+1}\dots P_{i,n}.$$



(a) (b)

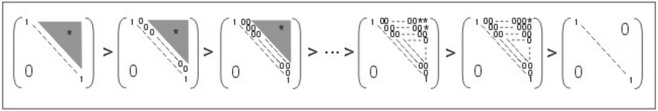
Characteristic subgroups

Lemma 1 (Levchuk) *Every characteristic subgroup $H \neq \{\mathbf{1}\}$ in the unitriangular group $UT_n(K)$ over the field K, $|K| > 2$, is a product of some subgroups P_{ij}, $i < j$, satisfying the following condition: if $P_{ij} \subseteq H$ then $Q_{ij} \subseteq H$.*

Verbal subgroups in $UT_n(K)$

Verbal subgroups in groups of unitriangular matrices over fields

Theorem 1 *Every verbal subgroup in the group $UT_n(K)$ over arbitrary field K coincides with one of the terms of the lower central series of this group.*



Problem: Given a set of words W, with which term of the lower central series does the verbal subgroup $V_W(UT_n(K))$ coincide?

Equalities on verbal subgroups in $UT_n(K)$

Outer-commutator words:

- outer commutator of weight 1: $w_1 = x_i, \quad i \in \mathbf{N}$
- outer commutator of weight n: $[u_r, v_{n-r}]$, where $u_r = u(x_1, ..., x_r)$ and $v_{n-r} = v(x_{r+1}, ..., x_n)$ are outer-commutator words of weight r and $n-r$ respectively

Theorem 2 *For an arbitrary field K, $|K| > 2$ and an outer-commutator word w_k of weight k we have:*

$$V_{w_k}(UT_n(K)) = V_{c_k}(UT_n(K)).$$

Equalities on verbal subgroups in $UT_n(K)$

Theorem 3 *For every field K, $|K| > 2$ and a power word x^m, where $m \in \mathbb{Z} \setminus \{0\}$ the following equalities hold:*

1. *If char$K = 0$ then $V_{x^m}(UT_n(K)) = UT_n(K)$;*
2. *If char$K = p$ and $LCD(m, p) = 1$ then*

$$V_{x^m}(UT_n(K)) = UT_n(K);$$

3. *If char$K = p$ then for every $k \in \mathbb{N}$ and $r \in \mathbb{Z}$ such that $LCD(r, p) = 1$ we have*

$$V_{x^{p^k \cdot r}}(UT_n(K)) = V_{c_{p^k}}(UT_n(K)).$$

Width of verbal subgroups in groups of unitriangular matrices

Width of verbal subgroups in $UT_n(K)$

Theorem 4 *Let* $\operatorname{char} K = 0$. *Then*

1. $wid_{w_k}(UT_n(K)) = 1$ *for every outer commutator* w_k *of weight* k*,*
2. $wid_{x^k}(UT_n(K)) = 1$ *for every integer number* k*.*

Width of verbal subgroups in $UT_n(K)$

Theorem 5 *Let* $\mathrm{char} K = p > 0$. *Then*

1. $wid_{w_k}(UT_n(K)) = 1$ *for every outer commutator* w_k *of weight* k,
2. *If* k *is an integer number, then*

$$wid_{x^k}(UT_n(K)) = \begin{cases} 1 & p \not| k \\ 2 & \text{otherwise} \end{cases}$$

Thank you.

For further reading

1. Bier A., "*Verbal subgroups in the group of triangular matrices over field of characteristic 0*", J. Algebra 321, 2 (2009), 483-494
2. Bier A., "*The width of verbal subgroups in the group of unitriangular matrices over a field*", in preparation
3. Levchuk V. M., "*Subgroups of the unitriangular group*", Izv. Ross. Akad. Nauk Ser. Mat., 38 (1974), p.1202-1220.
4. Segal, D., "*Words. Notes on Verbal Width in Groups.*", London Math. Soc. Lecture Note Ser., vol. 361, Cambridge University Press, Cambridge, 2009