THEOREMS OF SYLOW TYPE

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Groups and their actions, Będlewo 2010, 25th August

In 1872, the Norwegian mathematician L. Sylow proved the following theorem.

Theorem (L. Sylow)

Let G be a finite group and let $|G| = p^\alpha m$, where p is a prime and $(p, m) = 1$. Then

- $\mathcal{E}$ *G possesses a subgroup of order p^α (the so-called Sylow p-subgroup);*
- $\mathcal{C}$ *every two Sylow p-subgroups are conjugate;*
- $\mathcal{D}$ *every p-subgroup of G is included in a Sylow p-subgroup of G.*

In this talk we will discuss theorems of Sylow type, i.e. statements that resemble and generalize Sylow's theorem for an appropriate class of finite groups.

Recall the notion of π-Hall subgroups which is a natural generalization of the concept of Sylow p-subgroups and make some conventions about the notation and terminology.

By π we always denote a set of primes,

π' is the complement of π in the set of all primes.

By $\pi(n)$ we denote the set of prime divisors of an integer n.

An integer n is called a π-number, if $\pi(n) \subseteq \pi$.

For a finite group G, we set $\pi(G) = \pi(|G|)$.

G is a π-*group* if $|G|$ is a π-number.

A subgroup H of G is called a π-*Hall subgroup* if $\pi(H) \subseteq \pi$ and $\pi(|G : H|) \subseteq \pi'$.

The set of all π-Hall subgroups of G is denoted by $\mathrm{Hall}_\pi(G)$ (note that this set may be empty).

If the order and the index of a subgroup H of a finite group G are coprime (i.e. $H \in \mathrm{Hall}_\pi(G)$ for an appropriate set π) then H is simply called a Hall subgroup.

So, if $\pi = \{p\}$ then every π-Hall subgroup of G is a Sylow p-subgroup and vice versa.

Definition

Following P. Hall, we say that a finite group G *satisfies*

- $\mathcal{E}_\pi$ if $\mathrm{Hall}_\pi(G) \neq \varnothing$ (i.e., there exists a π-Hall subgroup in G);
- $\mathcal{C}_\pi$ if G satisfies $\mathcal{E}_\pi$ and every two π-Hall subgroups of G are conjugate;
- $\mathcal{D}_\pi$ if G satisfies $\mathcal{C}_\pi$ and every π-subgroup of G is included in a π-Hall subgroup.

A group G satisfying $\mathcal{E}_\pi$ (resp. $\mathcal{C}_\pi$, $\mathcal{D}_\pi$) is called an $\mathcal{E}_\pi$- (resp. $\mathcal{C}_\pi$-, $\mathcal{D}_\pi$-) *group*.

Given a set of primes π, we also denote by $\mathcal{E}_\pi$, $\mathcal{C}_\pi$, and $\mathcal{D}_\pi$ the classes of all finite $\mathcal{E}_\pi$-, $\mathcal{C}_\pi$-, and $\mathcal{D}_\pi$- groups, respectively.

Thus $G \in \mathcal{D}_\pi$ if the complete analogue of Sylow's theorem for the π-subgroups holds for G (we will also say that *the π-Sylow theorem holds for G*),
while $G \in \mathcal{C}_\pi$ and $G \in \mathcal{E}_\pi$ if weaker analogues of Sylow's theorem hold.

Theorem (P.Hall, 1928)

If a finite group G is solvable then $G \in \mathcal{D}_\pi$ for every set π of primes.

In contrast with Sylow's and Hall's theorems, there exists a set of primes π and a finite group G such that $\mathrm{Hall}_\pi(G) = \varnothing$. There are examples showing that, in general, $\mathcal{E}_\pi \neq \mathcal{C}_\pi$ and $\mathcal{C}_\pi \neq \mathcal{D}_\pi$.

Alt_5 does not possess a subgroup of order 15, hence $\mathrm{Alt}_5 \notin \mathcal{E}_{\{3,5\}}$.

There are exactly two classes of conjugate $\{2,3\}$-Hall subgroups of $\mathrm{GL}_3(2)$: the stabilizers of lines and planes, respectively. So $\mathrm{GL}_3(2) \in \mathcal{E}_{\{2,3\}} \setminus \mathcal{C}_{\{2,3\}}$.

Every subgroup of order 12 of Alt_5 is a point stabilizer, and all point stabilizers are conjugate. On the other hand, Alt_5 includes a $\{2,3\}$-subgroup $\langle (123), (12)(45) \rangle \simeq \mathrm{Sym}_3$ which acts without fixed points. Therefore, $\mathrm{Alt}_5 \in \mathcal{C}_{\{2,3\}} \setminus \mathcal{D}_{\{2,3\}}$.

Later P.Hall and, independently, S.A.Chunikhin proved the converse statement of Hall's theorem. Their results can be summarized in

Theorem (P.Hall and S.A.Chunikhin, 20-30s)

Let G be a finite group. The following statements are equivalent:

1. *G is solvable;*
2. *$G \in \mathcal{D}_\pi$ for every set π of primes;*
3. *$G \in \mathcal{E}_{p'}$ for every prime p.*

If we fix π, then the classes $\mathcal{E}_\pi, \mathcal{C}_\pi, \mathcal{D}_\pi$ can be wider than the class of solvable finite groups. It is clear, for example, that each π- or π'- group satisfies $\mathcal{D}_\pi$.

In this talk we consider the following

General Problem

Given a set π of primes and a finite group G, does G satisfy $\mathcal{E}_\pi, \mathcal{C}_\pi$ or $\mathcal{D}_\pi$?

Proposition

If $A \trianglelefteq G$ *and* $H \in \mathrm{Hall}_\pi(G)$ *then* $HA/A \in \mathrm{Hall}_\pi(G/A)$ *and* $H \cap A \in \mathrm{Hall}_\pi(A)$.

$$|G/A : HA/A||A : (H \cap A)| = \frac{|G|}{|HA|}\frac{|HA|}{|H|} = |G : H| \Rightarrow$$

$$\pi(|G/A : HA/A|) \cup \pi(|A : (H \cap A)|) \subseteq \pi(|G : H|) \subseteq \pi',$$
$$\text{while} \quad \pi(HA/A) \cup \pi(H \cap A) \subseteq \pi(H) \subseteq \pi.$$

Corollary

If $A \trianglelefteq G$ *and* $G \in \mathcal{E}_\pi$ *then* $G/A \in \mathcal{E}_\pi$ *and* $A \in \mathcal{E}_\pi$.

Corollary

If $G \in \mathcal{E}_\pi$, *then every composition factor of* G *satisfies* $\mathcal{E}_\pi$.

We see that the condition $\mathcal{E}_\pi$ and Hall subgroups of a finite group G are closely related to the normal and composition structures of G.

Problem

Describe the Hall subgroups in the finite simple groups.

This quite difficult problem was solved by:

P.Hall and J.Thompson for the symmetric and alternating groups;

E.Spitznagel, F.Gross, E.P.Vdovin and D.R. for the groups of Lie type and the sporadic groups.

Theorem (mod CFSG)

The Hall subgroups of every finite simple group are known.

Unfortunately, the description of the Hall subgroups in the finite simple groups is rather complicated, and we will not consider it here. Let us discuss some consequences of this description. In particular, how can one use this description to obtain criteria for $\mathcal{E}_\pi$, $\mathcal{C}_\pi$ and $\mathcal{D}_\pi$?

Given a group G, we want to understand whether or not G satisfies a condition $\Phi \in \{\mathcal{E}_\pi, \mathcal{C}_\pi, \mathcal{D}_\pi\}$. In view of above, we may expect that Φ depends on the normal and composition structure of G.

In the ideal situation, it would be so:

1. $G \in \Phi$ iff every composition factor of G satisfies Φ.
2. The simple groups satisfying Φ are known.

Let us understand, whether this ideal situation holds for $\mathcal{E}_\pi$, $\mathcal{C}_\pi$ and $\mathcal{D}_\pi$, i.e. whether these classes are closed under taking

1. the normal subgroups,
2. the homomorphic images, and
3. the extensions.

Is the class $\Phi \in \{\mathcal{E}_\pi, \mathcal{C}_\pi, \mathcal{D}_\pi\}$ closed under taking the normal subgroups, the homomorphic images and the extensions?

Φ	$\mathcal{E}_\pi$	$\mathcal{C}_\pi$	$\mathcal{D}_\pi$
normal subgroups			
homomorphisms			
extensions			

Is the class $\Phi \in \{\mathcal{E}_\pi, \mathcal{C}_\pi, \mathcal{D}_\pi\}$ closed under taking the normal subgroups, the homomorphic images and the extensions?

Φ	$\mathcal{E}_\pi$	$\mathcal{C}_\pi$	$\mathcal{D}_\pi$
normal subgroups	Yes		
homomorphisms			
extensions			

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Φ	$\mathcal{E}_\pi$	$\mathcal{C}_\pi$	$\mathcal{D}_\pi$
normal subgroups	Yes		
homomorphisms	Yes		
extensions			

Proposition (S.A.Chunikhin, about 1950)

Let A be a normal subgroup of G. If both A and G/A satisfy $\mathcal{C}_\pi$, then $G \in \mathcal{C}_\pi$.

Is the class $\Phi \in \{\mathcal{E}_\pi, \mathcal{C}_\pi, \mathcal{D}_\pi\}$ closed under taking the normal subgroups, the homomorphic images and the extensions?

Φ	$\mathcal{E}_\pi$	$\mathcal{C}_\pi$	$\mathcal{D}_\pi$
normal subgroups	Yes		
homomorphisms	Yes		
extensions			

Is the class $\Phi \in \{\mathcal{E}_\pi, \mathcal{C}_\pi, \mathcal{D}_\pi\}$ closed under taking the normal subgroups, the homomorphic images and the extensions?

Φ	$\mathcal{E}_\pi$	$\mathcal{C}_\pi$	$\mathcal{D}_\pi$
normal subgroups	Yes		
homomorphisms	Yes		
extensions		Yes	

Proposition

Let A be a normal subgroup of G.

- *Every π-subgroup of G/A is an image under the natural homomorphism of a π-subgroup of G.*
- *If $G \in \mathcal{D}_\pi$ then $G/A \in \mathcal{D}_\pi$.*

Is the class $\Phi \in \{\mathcal{E}_\pi, \mathcal{C}_\pi, \mathcal{D}_\pi\}$ closed under taking the normal subgroups, the homomorphic images and the extensions?

Φ	$\mathcal{E}_\pi$	$\mathcal{C}_\pi$	$\mathcal{D}_\pi$
normal subgroups	Yes		
homomorphisms	Yes		
extensions		Yes	

Is the class $\Phi \in \{\mathcal{E}_\pi, \mathcal{C}_\pi, \mathcal{D}_\pi\}$ closed under taking the normal subgroups, the homomorphic images and the extensions?

Φ	$\mathcal{E}_\pi$	$\mathcal{C}_\pi$	$\mathcal{D}_\pi$
normal subgroups	Yes		
homomorphisms	Yes		Yes
extensions		Yes	

We need the following

Extension Lemma

If $A \trianglelefteq G$, $\pi(G/A) \subseteq \pi$, and $M \in \mathrm{Hall}_\pi(A)$, then there exists $H \in \mathrm{Hall}_\pi(G)$ with $H \cap A = M$ if and only if G, acting by conjugation, leaves invariant the conjugacy class $M^A = \{M^a \mid a \in A\}$.

Let $\pi = \{2, 3\}$ and let $G = \mathrm{GL}_3(2) \cong \mathrm{PSL}_3(2)$ be a group of order $168 = 2^3 \cdot 3 \cdot 7$. Then G has exactly two classes of conjugate π-Hall subgroups with the representatives

$$\begin{pmatrix} \boxed{\mathrm{GL}_2(2)} & * \\ & \boxed{1} \end{pmatrix} \text{ and } \begin{pmatrix} \boxed{1} & * \\ & \boxed{\mathrm{GL}_2(2)} \end{pmatrix}.$$

The first one consists of the line stabilizers in the natural representation of G, and the second one consists of the plane stabilizers. The map

$$\iota : x \mapsto (x^t)^{-1}, \quad x \in G,$$

is an automorphism of G of order 2 (here x^t is the transpose of x). This map interchanges the classes of π-Hall subgroups, hence by the Extension Lemma the group $\widehat{G} = G : \langle \iota \rangle$ does not possess a π-Hall subgroup.

This example shows that $\mathcal{E}_\pi$ is not closed under extensions.

Is the class $\Phi \in \{\mathcal{E}_\pi, \mathcal{C}_\pi, \mathcal{D}_\pi\}$ closed under taking the normal subgroups, the homomorphic images and the extensions?

Φ	$\mathcal{E}_\pi$	$\mathcal{C}_\pi$	$\mathcal{D}_\pi$
normal subgroups	Yes		
homomorphisms	Yes		Yes
extensions		Yes	

Is the class $\Phi \in \{\mathcal{E}_\pi, \mathcal{C}_\pi, \mathcal{D}_\pi\}$ closed under taking the normal subgroups, the homomorphic images and the extensions?

Φ	$\mathcal{E}_\pi$	$\mathcal{C}_\pi$	$\mathcal{D}_\pi$
normal subgroups	Yes		
homomorphisms	Yes		Yes
extensions	No	Yes	

Suppose $\pi = \{2, 3\}$. Let $G = \mathrm{GL}_5(2)$. Consider the automorphism

$$\iota : x \mapsto (x^t)^{-1}, \quad x \in G,$$

and the natural semidirect product $\widehat{G} = G : \langle \iota \rangle$. One can show that $G \in \mathcal{E}_\pi$ and, for the natural module V of G, each π-Hall subgroup is the stabilizer in G of a series

$$\{0\} = V_0 < V_1 < V_2 < V_3 = V$$

such that $\dim V_k/V_{k-1} \in \{1, 2\}$ for each $k = 1, 2, 3$. Therefore, there exist exactly three classes of conjugate π-Hall subgroups of G with the representatives H_1, H_2 and H_3 of the kinds

$$\begin{pmatrix} \boxed{\mathrm{GL}_2(2)} & & * \\ & \boxed{1} & \\ & & \boxed{\mathrm{GL}_2(2)} \end{pmatrix}, \begin{pmatrix} \boxed{1} & & * \\ & \boxed{\mathrm{GL}_2(2)} & \\ & & \boxed{\mathrm{GL}_2(2)} \end{pmatrix}, \begin{pmatrix} \boxed{\mathrm{GL}_2(2)} & & * \\ & \boxed{\mathrm{GL}_2(2)} & \\ & & \boxed{1} \end{pmatrix},$$

respectively. For each π-Hall subgroup H of $\widehat{G}$, the intersection $H \cap G$ is conjugate with one of H_1, H_2, H_3. The class H_1^G is ι-invariant, so by the Extension Lemma there exists a π-Hall subgroup H of $\widehat{G}$ with $H \cap G = H_1$. Furthermore, ι interchanges H_2^G and H_3^G. Thus H_2 and H_3 are not included in π-Hall subgroups of $\widehat{G}$.

Therefore, $\widehat{G}$ has exactly one class of conjugate π-Hall subgroups.

So $\widehat{G} \in \mathcal{C}_\pi$, while its normal subgroup G does not satisfy $\mathcal{C}_\pi$.

This example also shows that, for a normal subgroup A of an $\mathcal{E}_\pi$-group G, the map

$$\mathrm{Hall}_\pi(G) \to \mathrm{Hall}_\pi(A)$$

given by

$$H \mapsto H \cap A, \quad H \in \mathrm{Hall}_\pi(G)$$

is not surjective in general.

Is the class $\Phi \in \{\mathcal{E}_\pi, \mathcal{C}_\pi, \mathcal{D}_\pi\}$ closed under taking the normal subgroups, the homomorphic images and the extensions?

Φ	$\mathcal{E}_\pi$	$\mathcal{C}_\pi$	$\mathcal{D}_\pi$
normal subgroups	Yes		
homomorphisms	Yes		Yes
extensions	No	Yes	

Is the class $\Phi \in \{\mathcal{E}_\pi, \mathcal{C}_\pi, \mathcal{D}_\pi\}$ closed under taking the normal subgroups, the homomorphic images and the extensions?

Φ	$\mathcal{E}_\pi$	$\mathcal{C}_\pi$	$\mathcal{D}_\pi$
normal subgroups	Yes	No	
homomorphisms	Yes		Yes
extensions	No	Yes	

Is $\mathcal{D}_\pi$ (the class of finite groups in which the π-Sylow theorem holds) closed under taking normal subgroups and extensions?

H.Wielandt, in the survey talk given at the XIII International mathematical congress in Edinburgh in 1958, proposed the following problem:

Problem, H. Wielandt, 1958, Kourovka Notebook, Problems 3.62 and 13.33

Let A be a normal subgroup of G.

1. Does G satisfy $\mathcal{D}_\pi$ if both A and G/A satisfy $\mathcal{D}_\pi$?
2. Does A satisfy $\mathcal{D}_\pi$ if G satisfies $\mathcal{D}_\pi$?

The classification of the Hall subgroups in the finite simple groups allows us to observe the following statement:

Class Number Theorem (mod CFSG)

Let S be a finite simple $\mathcal{E}_\pi$-group and let $k_\pi(S)$ be the number of classes of conjugate π-Hall subgroups of S. Then the following statements hold:

- *if $2 \notin \pi$, then $k_\pi(S) = 1$;*
- *if $3 \notin \pi$, then $k_\pi(S) \in \{1, 2\}$;*
- *if $2, 3 \in \pi$, then $k_\pi(S) \in \{1, 2, 3, 4, 9\}$.*

In particular, $k_\pi(S)$ is a bounded π-number.

This observation has many fruitful consequences. In particular, the following theorem holds.

Theorem (E.P.Vdovin and D.R., mod CFSG)

If $G \in \mathcal{D}_\pi$ and $A \trianglelefteq G$ then $A \in \mathcal{D}_\pi$.

Proof of the Theorem $A \trianglelefteq G \in \mathcal{D}_\pi \Rightarrow A \in \mathcal{D}_\pi$:

Obviously, $A \in \mathcal{E}_\pi$ and every π-subgroup U of A is included in the π-Hall subgroup $H \cap A$ of A where H is a π-Hall subgroup of G such that $U \leq H$. Hence it is sufficient to prove that $A \in \mathcal{C}_\pi$. Let G be a minimal counterexample and A be a minimal normal subgroup of G with $A \notin \mathcal{C}_\pi$. Since $\mathcal{C}_\pi$ is closed under extensions, A is an ordinary minimal normal subgroup of G and

$$A = S_1 \times \cdots \times S_n,$$

where $S_1, \ldots, S_n$ are conjugate in G simple subgroups. Evidently, the number $k_\pi(A)$ of classes of conjugate π-Hall subgroups of A is equal to $k_\pi(S_1)^n$ and, by the Class Number Theorem, is a π-number. On the other hand, G acts on the set of these classes and this action is transitive, since $G \in \mathcal{D}_\pi$ and every π-Hall subgroup of A is of type $H \cap A$ for a π-Hall subgroup H of G. Hence, G has a unique orbit and the cardinality of this orbit equals $k_\pi(A)$. It is clear that $H \in \mathrm{Hall}_\pi(G)$ normalizes $H \cap A$ and stabilizes the class $(H \cap A)^A$. Thus, $k_\pi(A)$ is equal to the index of the stabilizer of this class and divides $|G : H|$. Therefore $k_\pi(A)$ is a π'-number. This means that $k_\pi(A) = 1$, i.e. $A \in \mathcal{C}_\pi$. A contradiction.

Thus the class $\mathcal{D}_\pi$ is closed under taking normal subgroups.

Is the class $\mathcal{D}_\pi$ closed under extensions? This problem has a rich history. Besides Wieland's talk, this problem is mentioned in M.Suzuki's and L.A.Shemetkov's books and it was recorded in Kourovka Notebook as the problem 3.62. First partial answer to the problem was given by the well-known Theorem D5 by P.Hall:

Theorem D5 (P.Hall, 1956)

Let A be a normal subgroup of G. If $A, G/A \in \mathcal{D}_\pi$, the π-Hall subgroups of A are nilpotent and the π-Hall subgroups of G/A are solvable then $G \in \mathcal{D}_\pi$.

Later the problem was investigated in particular cases by H.Wielandt, B.Hartley, L.A.Shemetkov, L.S.Kazarin and others.

Theorem (V.D.Mazurov, D.R., 1997)

The class $\mathcal{D}_\pi$ is closed under taking extensions iff $\mathrm{Aut}(S) \in \mathcal{D}_\pi$ *for every simple group* $S \in \mathcal{D}_\pi$.

In several papers during 1997-2008, E.Vdovin and D.R. checked that $\mathrm{Aut}(S) \in \mathcal{D}_\pi$ for every simple $\mathcal{D}_\pi$-group S and described the finite simple groups satisfying $\mathcal{D}_\pi$. It was quite a difficult problem, since to solve it one must know not only the π-Hall subgroups but also all π-subgroups of simple groups.

Theorem (V.Mazurov, E. Vdovin, D.R., 2006, mod CFSG)

Let A be a normal subgroup of G. Then $G \in \mathcal{D}_\pi$ iff $A, G/A \in \mathcal{D}_\pi$.

Is the class $\Phi \in \{\mathcal{E}_\pi, \mathcal{C}_\pi, \mathcal{D}_\pi\}$ closed under taking the normal subgroups, the homomorphic images and the extensions?

Φ	$\mathcal{E}_\pi$	$\mathcal{C}_\pi$	$\mathcal{D}_\pi$
normal subgroups	Yes	No	
homomorphisms	Yes		Yes
extensions	No	Yes	

Is the class $\Phi \in \{\mathcal{E}_\pi, \mathcal{C}_\pi, \mathcal{D}_\pi\}$ closed under taking the normal subgroups, the homomorphic images and the extensions?

Φ	$\mathcal{E}_\pi$	$\mathcal{C}_\pi$	$\mathcal{D}_\pi$
normal subgroups	Yes	No	Yes (mod CFSG)
homomorphisms	Yes		Yes
extensions	No	Yes	

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Φ	$\mathcal{E}_\pi$	$\mathcal{C}_\pi$	$\mathcal{D}_\pi$
normal subgroups	Yes	No	Yes (mod CFSG)
homomorphisms	Yes		Yes
extensions	No	Yes	Yes (mod CFSG)

Thus, we have the ideal situation for $\mathcal{D}_\pi$:

- A finite group G satisfies $\mathcal{D}_\pi$ iff every composition factor of G satisfies $\mathcal{D}_\pi$.
- The simple groups satisfying $\mathcal{D}_\pi$ are known.

What does "known" mean? It means that, for every finite simple group S, we have $S \in \mathcal{D}_\pi$ iff a number of arithmetic conditions hold for primes in π and the natural arithmetic arguments of S.

For instance, consider the situation where $2 \in \pi$ and $S = \mathrm{PSL}_n(q)$. Then $S \in \mathcal{D}_\pi$ iff one of the following statements hold:

- $\pi(S) \subseteq \pi$;
- $|\pi \cap \pi(S)| \leq 1$;
- q is odd, $\pi \cap \pi(S) \subseteq \pi(q - \varepsilon)$ where $\varepsilon = \pm 1$ and $q \equiv \varepsilon \pmod 4$, $s > n$ for every odd $s \in \pi \cap \pi(S)$ and $t > n + 1$ for every Fermat prime $t \in \pi \cap \pi(S)$.

A similar description is obtained in the all possible cases.

Thus, for every finite group G with known composition factors one can say whether or not G satisfies $\mathcal{D}_\pi$.

What applications does this result have?

For a finite group G, we denote by $\mathcal{O}_\pi(G)$ the π-radical of G, i. e. the largest normal π-subgroup of G.

Theorem (R. Baer, 1967, M. Suzuki, 1968)

Let G be a finite group and let p be a prime. Assume x is an element of G such that $\langle x, x^g \rangle$ is a p-group for every $g \in G$. Then $x \in \mathcal{O}_p(G)$.

Definition

We say that *the π-Baer-Suzuki theorem holds for a finite group G* and write $G \in \mathcal{BS}_\pi$, if $D \subseteq \mathcal{O}_\pi(G)$ for every conjugacy class D of G such that every two elements of D generate a π-subgroup.

Theorem $\mathcal{D}_\pi \subseteq \mathcal{BS}_\pi$ (unpublished, mod CFSG)

If the π-Sylow theorem holds for a group G then the π-Baer-Suzuki theorem holds for G.

In the particular case $\pi = \{p\}$ this theorem turns into the classical Baer-Suzuki theorem.

It is possible to investigate the question about the strictness of the inclusion $\mathcal{D}_\pi \subseteq \mathcal{BS}_\pi$.

Proposition (unpublished, mod CFSG)

Let $\mathcal{G}$ be the class of all finite groups. Then the following statements are equivalent:

1. $\mathcal{D}_\pi = \mathcal{BS}_\pi$;
2. $\mathcal{D}_\pi = \mathcal{G}$;
3. *either $|\pi| \leq 1$, or π is the set of all primes.*

Let us show that the class $\mathcal{BS}_\pi$ is smaller than the class $\mathcal{G}$. Moreover, in the inclusion $\mathcal{D}_\pi \subseteq \mathcal{BS}_\pi$, one cannot replace $\mathcal{D}_\pi$ with $\mathcal{C}_\pi$:

Let $p > 3$ be a prime and let $G = \mathrm{Sym}_p$. Assume $\pi = p'$. Then $\mathrm{Sym}_{p-1} \in \mathrm{Hall}_\pi(G)$ and G has a unique class of conjugate π-Hall subgroups, i.e. $G \in \mathcal{C}_\pi$. On the other hand, every $m < p - 1$ transpositions in G (in particular, every two transpositions) generate a π-subgroup, while $O_\pi(G) = 1$. Thus $G \notin \mathcal{BS}_\pi$.

We obtained a generalization of the Baer-Suzuki theorem by proving a π-analogue of the Baer-Suzuki theorem for the groups in $\mathcal{D}_\pi$. This may be a way to obtain generalizations of other "p-theorems".

Theorem (V.I. Zenkov, 1997, mod CFSG)

In every finite group G, there exist three Sylow p-subgroups P_1, P_2, and P_3 such that

$$P_1 \cap P_2 \cap P_3 = O_p(G).$$

Open problem

Does the π-Sylow theorem imply "the π-Zenkov theorem"? In other words, if $G \in \mathcal{D}_\pi$, do there exist three π-Hall subgroups H_1, H_2, and H_3 of G such that

$$H_1 \cap H_2 \cap H_3 = O_\pi(G)?$$

Let us return to the investigation of the classes $\mathcal{E}_\pi$ and $\mathcal{C}_\pi$. Although these classes are not closed under taking extensions and normal subgroups, respectively, the situation is not too dramatic.

Theorem (F.Gross, 1987, mod CFSG)

If π is a set of primes and $2 \notin \pi$ then $\mathcal{E}_\pi = \mathcal{C}_\pi$.

This result implies that if $2 \notin \pi$ then the class $\mathcal{E}_\pi = \mathcal{C}_\pi$ is closed under taking normal subgroups, homomorphic images and extensions. But even if $2 \in \pi$ then one can obtain a criterion for $\mathcal{E}_\pi$ in terms of the so-called G-induced automorphisms groups of the composition factors of G.

Definition

Suppose A, B, H are subgroups of G such that $B \trianglelefteq A$.
Then $N_H(A/B) = N_H(A) \cap N_H(B)$ is the *normalizer* of A/B in H.
If $x \in N_H(A/B)$ then x induces an automorphism of A/B by

$$Ba \mapsto Bx^{-1}ax.$$

Thus there exists a homomorphism $N_H(A/B) \to \mathrm{Aut}(A/B)$.
The image of $N_H(A/B)$ under this homomorphism is denoted by $\mathrm{Aut}_H(A/B)$ and is called the *group of H-induced automorphisms* of A/B.

In 1986, F.Gross obtained a sufficient condition for a finite group G to satisfy $\mathcal{E}_\pi$.

Theorem (F.Gross, 1986, mod CFSG)

Let $1 = G_0 < G_1 < \ldots < G_n = G$ be a composition series for a finite group G which is a refinement of a chief series for G. If $\mathrm{Aut}_G(G_i/G_{i-1}) \in \mathcal{E}_\pi$ for all $i = 1, \ldots, n$, then $G \in \mathcal{E}_\pi$.

By using the classification of the finite simple groups and the description of π-Hall subgroups in such groups the following theorem is obtained.

Theorem (E.Vdovin and D.R., 2010, mod CFSG)

Let $1 = G_0 < G_1 < \ldots < G_n = G$ be a composition series for a finite group G. If, for some i, $\mathrm{Aut}_G(G_i/G_{i-1}) \notin \mathcal{E}_\pi$, then $G \notin \mathcal{E}_\pi$.

Corollary ($\mathcal{E}_\pi$-criterion, mod CFSG)

Let $1 = G_0 < G_1 < \ldots < G_n = G$ be a composition series for a finite group G which is a refinement of a chief series of G. Then $G \in \mathcal{E}_\pi$ iff $\mathrm{Aut}_G(G_i/G_{i-1}) \in \mathcal{E}_\pi$ for all $i = 1, \ldots, n$.

We have mentioned that if A is a normal subgroup of an $\mathcal{E}_\pi$-group G, then there exist maps $\mathrm{Hall}_\pi(G) \to \mathrm{Hall}_\pi(G/A)$ and $\mathrm{Hall}_\pi(G) \to \mathrm{Hall}_\pi(A)$, given by $H \mapsto HA/A$ and $H \mapsto H \cap A$, respectively. It is surprising that the first map turns out to be surjective:

Corollary (mod CFSG)

Every π-Hall subgroup in a homomorphic image of an $\mathcal{E}_\pi$-group G is the image of a π-Hall subgroup of G.

The map $H \mapsto H \cap A$, in general, is not surjective, as we have seen above.

The previous corollary implies a statement which is related to the study of $\mathcal{C}_\pi$-groups.

Corollary (mod CFSG)

If $G \in \mathcal{C}_\pi$ and $A \trianglelefteq G$ then $G/A \in \mathcal{C}_\pi$.

We can fill in the last remaining position in our table.

Is the class $\Phi \in \{\mathcal{E}_\pi, \mathcal{C}_\pi, \mathcal{D}_\pi\}$ closed under taking the normal subgroups, the homomorphic images and the extensions?

Φ	$\mathcal{E}_\pi$	$\mathcal{C}_\pi$	$\mathcal{D}_\pi$
normal subgroups	Yes	No	Yes (mod CFSG)
homomorphisms	Yes		Yes
extensions	No	Yes	Yes (mod CFSG)

Is the class $\Phi \in \{\mathcal{E}_\pi, \mathcal{C}_\pi, \mathcal{D}_\pi\}$ closed under taking the normal subgroups, the homomorphic images and the extensions?

Φ	$\mathcal{E}_\pi$	$\mathcal{C}_\pi$	$\mathcal{D}_\pi$
normal subgroups	Yes	No	Yes (mod CFSG)
homomorphisms	Yes	Yes (mod CFSG)	Yes
extensions	No	Yes	Yes (mod CFSG)

This table reminds that $\mathcal{C}_\pi$ is not closed under subgroups and even normal subgroups. However, it is possible to show that some subgroups and normal subgroups of $\mathcal{C}_\pi$-groups satisfy $\mathcal{C}_\pi$. Moreover, it is possible to obtain a criterion for $\mathcal{C}_\pi$. By using the Class Number Theorem one can prove

HA-Theorem (E.P. Vdovin and D.R., 2010, mod CFSG)

Let $G \in \mathcal{C}_\pi$, $H \in \mathrm{Hall}_\pi(G)$, and $A \trianglelefteq G$. Then $HA \in \mathcal{C}_\pi$.

Corollary ($\mathcal{C}_\pi$-criterion, mod CFSG)

Let $A \trianglelefteq G$. Then $G \in \mathcal{C}_\pi$ if and only if $G/A \in \mathcal{C}_\pi$ and, for every (equivalently, for some) intermediate subgroup $A \leq K \leq G$ such that $K/A \in \mathrm{Hall}_\pi(G/A)$, we have $K \in \mathcal{C}_\pi$.

Corollary (mod CFSG)

If $A \trianglelefteq G$ and $|G : A|$ is a π'-number, then $G \in \mathcal{C}_\pi \Leftrightarrow A \in \mathcal{C}_\pi$.

By using the $\mathcal{E}_\pi$- and C_π-criterion, one can show that the question whether a given group G satisfies $\mathcal{E}_\pi$ or $\mathcal{C}_\pi$, is reduced to the same question for several almost simple groups.

Recall that a finite group G is said to be *almost simple*, if

$$S \simeq \mathrm{Inn}(S) \leq G \leq \mathrm{Aut}(S)$$

for a nonabelian simple group S.

Open problem

Describe all finite almost simple groups satisfying $\mathcal{E}_\pi$ and $\mathcal{C}_\pi$.

Now the problem of describing almost simple $\mathcal{E}_\pi$- and $\mathcal{C}_\pi$-groups is close to being solved modulo the classification of π-Hall subgroups in simple groups.

In view of the Extension Lemma, in order to solve this problem it is required to study the action of $G \leq \mathrm{Aut}(S)$ on the set of classes π-Hall subgroups of S for every simple S.

Although the theory of $\mathcal{C}_\pi$-groups is not complete now, one can use the developed part of this theory to obtain new theorems. I wish to discuss some fresh results.

Evidently,

If $G \in \mathcal{E}_\pi$, $H \in \mathrm{Hall}_\pi(G)$, and $H \leq M \leq G$ then $M \in \mathcal{E}_\pi$.

It is surprising that one can replace $\mathcal{E}_\pi$ with $\mathcal{C}_\pi$ in this statement.
It is already known that if M is the product of H and a normal subgroup of G, one can replace $\mathcal{E}_\pi$ with $\mathcal{C}_\pi$:

HA-Theorem

Let $G \in \mathcal{C}_\pi$, $H \in \mathrm{Hall}_\pi(G)$, and $A \trianglelefteq G$. Then $HA \in \mathcal{C}_\pi$.

Furthermore, by using the classification of the Hall subgroups in simple groups, one can prove the following statement:

Theorem (unpublished, mod CFSG)

In the finite simple groups, the Hall subgroups are pronormal.

Recall that a subgroup H of G is called *pronormal* if, for every $g \in G$, the subgroups H and H^g are conjugate in $\langle H, H^g \rangle$.

By using the last result and the HA-Theorem, we proved

Theorem (unpublished, mod CFSG)

In $\mathcal{C}_\pi$-groups, every π-Hall subgroup is pronormal.

One can equivalently reformulate this statement:

Theorem (unpublished, mod CFSG)

Let $G \in \mathcal{C}_\pi$, $H \in \mathrm{Hall}_\pi(G)$, and $H \leq M \leq G$. Then $M \in \mathcal{C}_\pi$.

In other words, the class $\mathcal{C}_\pi$ is closed under taking overgroups of π-Hall subgroups.

Note that π-Hall subgroups are not pronormal in general.

Proposition

Suppose π is a set of primes with $\mathcal{C}_\pi \neq \mathcal{E}_\pi$. Let $X \in \mathcal{E}_\pi \setminus \mathcal{C}_\pi$ and $p \in \pi'$. Then $G = X \wr \mathbb{Z}_p$ possesses non-pronormal $H \in \mathrm{Hall}_\pi(G)$.

Open poblem

Conjecture: if $G \in \mathcal{D}_\pi$, $H \in \mathrm{Hall}_\pi(G)$, and $H \leq M \leq G$ then $M \in \mathcal{D}_\pi$.

Recall that a class $\mathcal{X}$ of finite groups is a *formation* if $\mathcal{X}$ is closed under taking homomorphic images and finite subdirect products, i.e.

- if $G \in \mathcal{X}$ and $A \trianglelefteq G$ then $G/A \in \mathcal{X}$ and
- if $A, B \trianglelefteq G$ and $G/A, G/B \in \mathcal{X}$ then $G/(A \cap B) \in \mathcal{X}$.

A formation $\mathcal{X}$ is said to be *saturated* if, for every $A \trianglelefteq G$ such that $A \leq \Phi(G)$ and $G/A \in \mathcal{X}$, we have $G \in \mathcal{X}$.

Theorem (E.P.Vdovin, L.A.Shemetkov and D.R., unpublished, mod CFSG)

The classes $\mathcal{E}_\pi$, $\mathcal{C}_\pi$, and $\mathcal{D}_\pi$ are saturated formations.

This statement gives an affirmative answer to the problems 18 and 19 in Shemetkov's book "Formations of finite groups" and the problem 5.65 in the Kourovka notebook.

Thank you for your attention!