ON SOME INFINITE DIMENSIONAL LINEAR GROUPS

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Let $F$ be a field and $A$ a vector space over $F$. Denote by $\text{GL}(F, A)$ the group of all $F$–automorphisms of $A$. The subgroups of $\text{GL}(F, A)$ are called the linear groups. Linear groups play a very important role in algebra and other branches of mathematics. If $\dim_F(A)$ (the dimension of $A$ over $F$) is finite, say $n$, then a subgroup $G$ of $\text{GL}(F, A)$ is a finite dimensional linear group. It is well known that in this case, $\text{GL}(F, A)$ can be identified with the group of all invertible $n \times n$ matrices with entries in $F$. The theory of finite dimensional linear groups is one of the best developed in group theory. It uses not only algebraic, but also topological, geometrical, combinatorial, and many other methods.

However, in the case when $A$ has infinite dimension over $F$, the study of the subgroups of $\text{GL}(F, A)$ requires some additional restrictions. This case is much more complicated and requires some additional restrictions allowing an effective employing of already developed techniques. The most natural restrictions here are the finiteness conditions. Finitary linear groups demonstrate the efficiency of such approach. We recall that a subgroup $G$ of $\text{GL}(F, A)$ is called finitary if, for every element $g \in G$, its centralizer $C_A(g)$ has finite codimension over $F$. The theory of finitary linear groups is now well-developed by many mathematicians and many interesting results have been proven there (see, e.g., the survey [1]). We begin with the consideration of some generalizations of such groups.

1. ON SOME GENERALIZATIONS OF FINITARY LINEAR GROUPS

If $G$ is a subgroup of $\text{GL}(F, A)$, then we can consider the vector space $A$ as a module over the group ring $FG$. We can obtain the following generalizations of finitary groups. At the transition from the field $F$ to the ring $R$, artinian and noetherian $R$–modules are natural generalizations of the concept of finite dimensional vector space. Some related generalizations of finitary groups have been considered by B.A.F. Wehrfritz (see [2–5]).

Let $R$ be a ring, $G$ be a group and $A$ be an $RG$–module. Following B.A.F. Wehrfritz, a group $G$ is called artinian–finitary, if for every element $g \in G$, the factor–module $A/C_A(g)$ is artinian as an $R$–module. In this case, we say that $A$ is artinian–finitary $RG$–module.
We observe that we can consider finitary linear groups as linear analogs of the FC groups (we can define FC–group $G$ as a group for whose $|G:C_G(x)|$ is finite for each element $g \in G$). Similarly, if $R = \mathbb{Z}$ and $G$ is an artinian–finitary group, then the additive group of the factor–module $A/C_A(g)$ is Chernikov for every element $g \in G$. This shows that we can consider artinian–finitary groups as linear analogs of the groups with Chernikov conjugacy classes (shortly CC–groups).

One of the first important results of theory of FC–groups was a theorem due to B.H. Neumann that described the structure of FC–groups with bounded conjugacy classes. Following B.H. Neumann, a group $G$ is called a BFC–group if there exists a positive integer $b$ such that $|g^G| \leq b$ for each element $g \in G$. B.H. Neumann proved that a group $G$ is a BFC–group if and only if the derived subgroup $[G,G]$ is finite [6, Theorem 3.1].

A group $G \leq \text{GL}(F, A)$ is said to be the bounded finitary linear group, if there is a positive integer $b$ such that $\dim_F(A/C_A(g)) \leq b$ for each element $g \in G$. These groups are some linear analogs of the BFC–groups. Let $\omega RG$ be the augmentation ideal of the group ring $RG$, the two–sided ideal of $RG$ generated by all elements $g-1$, $g \in G$. The submodule $A_\omega FG$ is called the derived submodule. We can consider the derived submodule as a linear analog of a derived subgroup. Note that in general case we cannot obtain an analog of a Neumann's theorem. It is not hard to construct an $F_p G$–module $A$ over an infinite elementary abelian group $G$ such that $G$ is bounded finitary linear group but $A(\omega A_\omega F_p G))$ has an infinite dimension over $F_p$ (see [7]). However, under some natural restrictions on the $p$–sections of a bounded finitary linear group, the finiteness of $\dim_F(A(\omega FG))$ can be proven; that is, some linear analog of B.H. Neumann’s theorem can be established. We considered it for a more general situation.

Let $A$ be an artinian $\mathbb{Z}$–module. Then a set $\Pi(A)$ is finite. If $D$ is a divisible part of $A$, then $D = K_1 \oplus \ldots \oplus K_d$ where $K_j$ is a Prüfer subgroup, $1 \leq j \leq d$. The number $d$ is an invariant of $A$. Another important invariant here is the order of $A/D$.

If $D$ is a Dedekind domain, then the structure of artinian $D$–module $A$ is very similar to the described above. Let $D$ be a Dedekind domain. Put

$$\text{Spec}(D) = \{ P \mid P \text{ is a maximal ideal of } D \}.$$ 

Let $P$ be a maximal ideal of $D$. Denote by $A_P$ the set of all elements $a$ such that $\text{Ann}_D(a) = P^n$ for some positive integer $n$. If $A$ is a $D$–periodic module, then define

$$\text{Ass}_D(A) = \{ P \in \text{Spec}(D) \mid A_P \neq 0 \}.$$ 

In this case, $A = \bigoplus_{P \in \pi} A_P$ where $\pi = \text{Ass}_D(A)$ (see, for example, [8, Corollary 6.25]). If $A$ is an artinian $D$–module, then $A$ is $D$–periodic and the set $\text{Ass}_D(A)$ is finite. Furthermore, $A = K_1 \oplus \ldots \oplus K_d \oplus B$ where $K_j$ is a Prüfer submodule, $1 \leq j \leq d$, $B$ is a finitely generated submodule (see, for example, [9, Theorem 5.7]). Here the Prüfer submodule is a $D$–injective envelope of a simple submodule. Observe that this decomposition is unique up to isomorphism. It follows that a number $d$ is an invariant of the module $A$. Put $d = I_P(A)$. The submodule $B$ has a finite series of submodules with $D$–simple factors. The Jordan–Hölder Theorem implies that the length of this composition series is also an invariant of $B$, and hence of $A$. Denote this number by $I_P(A)$.
Let $D$ be a Dedekind domain and $G$ be a group. The $DG$–module $A$ is said to be a bounded artinian finitary if $A$ is artinian finitary and there are the positive integers $b$ and a finite subset $\tau \subseteq \text{Spec}(D)$ such that $l_F(A/C_A(g)) \leq b$, $l_D(A/C_A(g)) \leq d$ and $\text{Ass}_D(A/C_A(g)) \subseteq b_\sigma(A)$. We will use the following notation:

$$\pi(A) = \{ p \mid p = \text{char} \ D/P \text{ for all } P \in b_\sigma(A) \}.$$ 

The group $G$ is said to be generalized radical if $G$ has an ascending series whose factors are locally nilpotent or locally finite. Let $p$ be a prime. We say that a group $G$ has finite section $p$–rank $r_p(G) = r$ if every elementary abelian $p$–section $U/V$ of $G$ is finite of order at most $p^r$ and there is an elementary abelian $p$–section $A/B$ of $G$ such that $|A/B| = p^r$.

In the paper [10], the following analog of the Neumann’s theorem has been obtained.

1.1. THEOREM (L.A. Kurdachenko, I.Ya. Subbotin, V.A. Chepurdya [10]). Let $D$ be a Dedekind domain, $G$ be a locally generalized radical group, and $A$ be a $DG$–module. Suppose that $A$ is a bounded artinian finitary module. Assume also that there exists a positive integer $r$ such that the section $p$–rank of $G$ is at most $r$ for all $p \in \pi(A)$. Then

(a) the submodule $A(\omega DG)$ is artinian as $D$–module,

(b) the factor–group $G/C_0(A)$ has a finite special rank.

1.2. COROLLARY. Let $F$ be a field, $A$ be a vector space over $F$, $G$ be a locally generalized radical subgroup of $GL(F, A)$. Suppose that there exists a positive integer $r$ such that the section $p$–rank of $G$ is at most $r$ where $p = \text{char} F$. Then

(a) the submodule $A(\omega FG)$ is finite dimensional,

(b) the factor–group $G/C_0(A)$ has a finite special rank.

As we noted above the restriction on the section $p$–rank is essential.

2. LINEAR GROUPS THAT ARE DUAL TO FINITARY

Consider another analog of FC–groups which are dual in some sense to finitary linear groups. We introduce this concept not only for linear groups, but in more general situation.

Let $R$ be a ring, $G$ be a group and $A$ be an $RG$–module. If $a$ is an element of $A$, then the set

$$aG = \{ ag \mid g \in G \}$$

is called the $G$–orbit of an element $a$.

We say that $G$ has finite orbits on $A$ if the orbits $aG$ are finite for all $a \in A$. 

By the orbit stabilizer theorem, it is clear that in this situation, \(|aG| = |G : C_G(a)|\) is finite, so we can think of \(aG\) as the analog of a conjugacy class.

Let \(F\) be a field and let \(G\) be a subgroup of \(\text{GL}(F, A)\). Suppose that \(\text{dim}_F (A)\) is finite and choose the basis \(a_1, \ldots, a_n\) of the vector space \(A\). Suppose that \(G\) has finite orbits on \(A\). Then every element of \(C_G(a_1) \cap \ldots \cap C_G(a_n)\) acts trivially on \(A\), and hence \(C_G(a_1) \cap \ldots \cap C_G(a_n) = \langle 1 \rangle\). However, this intersection has finite index in \(G\) and hence \(G\) is finite. Thus, we can think of linear groups with finite orbits as a generalization of finite groups.

We say that \(G\) has \textit{boundedly finite orbits on } \(A\) if there is a positive integer \(B\) such that \(|aG| \leq b\) for each element \(a \in A\). The smallest such \(b\) will be denoted by \(l_0A(G)\).

Since \(|aG| = |G : C_G(a)|\) for all \(a \in A\), it is not hard to see that a group \(G\) in which \(G/C_G(A)\) is finite is an example of a group with boundedly finite orbits on \(A\). However, as the following example shows, the converse statement is far from being true.

Let \(A\) be a vector space over the field \(F\) admitting the basis \(\{a_n| n \in \mathbb{N}\}\). For every \(n \in \mathbb{N}\) the mapping \(g_n: A \rightarrow A\), given by

\[
\begin{align*}
g_n(a) &= \begin{cases} 
a_1 + a_m & \text{if } m = n + 1 \\
a_m & \text{if } m \neq n + 1
\end{cases}
\end{align*}
\]

is an \(F\)–automorphism of \(A\). Let \(G = \langle g_n | n \in \mathbb{N} \rangle\), then \(G\) is a subgroup of \(\text{GL}(F, A)\). Clearly \([g_n, g_m] = 1\) whenever \(n \neq m\), so that \(G\) is abelian. Moreover, if \(\text{char} F = p > 0\), then \(G\) is an elementary abelian \(p\)–group. It follows in this case that \(ag = a + ta_1\) for every \(a \in A\), where \(0 \leq t < p\). Consequently,

\[
aG = \{a, a + a_1, a + 2a_1, \ldots, a + (p-1)a_1\}.
\]

Therefore, \(|aG| \leq p\) for each element \(a \in A\), and \(G\) has boundedly finite orbits on \(A\). However, it is clear that \(C_G(A) = \langle 1 \rangle\), so that \(G/C_G(A)\) is infinite.

Let \(B\) be a vector space over a field \(F\) of characteristic \(p > 0\) admitting the basis \(\{b_n| n \in \mathbb{N}\}\). We define the mapping \(x: B \rightarrow B\) by the rule

\[
\begin{align*}
x(b) &= \begin{cases} 
b_m & \text{if } m \text{ is even} \\
b_{2n} + b_{2n+1} & \text{if } m = 2n + 1
\end{cases}
\end{align*}
\]

Clearly, \(x\) is an \(F\)–automorphism of \(B\) and \(B(\omega F < x >) = \bigoplus_{n \in \mathbb{N}} b_{2n} F\). In particular, the dimension of \(B(\omega F < x >)\) is infinite. Since \(|x| = p, |b < x >| \leq p\) for each element \(b \in B\). Now let \(A\) and \(G\) be the vector space and the linear group from the first example. Then \(L = G \times < x >\) acts on the vector space \(C = A \oplus B\) in the natural way. Clearly, \(|cL| \leq p^2\) for every element \(c \in C\). However, the factor–group \(L/C_L(C)\) is infinite and the dimension of \(C(\omega FL)\) is infinite. In other words, we cannot have an analog of Neumann’s theorem.

Next result describes the linear groups acting with boundedly finite orbits.
2.1. **THEOREM** (M.R. Dixon, L.A. Kurdachenko, J. Otal [11]). Let $G$ be a group, $R$ be a ring and $A$ be an $RG$–module. Suppose that $G$ acts on $A$ with boundedly finite $G$–orbits, and let $b = \text{lo}_A(G)$. Then

(i) $G/C_G(A)$ includes a normal abelian subgroup $L/C_G(A)$ of finite exponent such that $G/L$ is finite.

(ii) $A$ includes an $RG$–submodule $C$ such that $C$ is finitely generated as an $R$–module and $L$ acts trivially on $C$ and $A/C$.

(iii) There is a positive integer $m$ such that $m$ is a divisor of $b!$ and $mA(\omega RG) = <0>$.

We note that for our ring $R$ in Theorem 2.1, even though $C$ is a finitely generated $R$–submodule, its submodules need not be finitely generated. Therefore, we cannot deduce in this theorem that $A(\omega RG)$ is finitely generated as an $R$–module. However, if $R$ is noetherian, then every finitely generated $R$–submodule is also noetherian so in this case, every submodule of $C$ is finitely generated. Even when $R$ is a noetherian ring so that $A(\omega RG)$ is finitely generated, in general it appears that nothing can be deduced concerning its number of generators. We can now establish our next main theorem.


(i) Suppose that $G$ acts on $A$ with boundedly finite $G$–orbits, and let $b = \text{lo}_A(G)$. Then $G/C_G(A)$ includes a normal abelian subgroup $L/C_G(A)$ of finite index such that $A(\omega RG)$ is finitely generated.

(ii) If a factor–group $G/C_G(A)$ has a normal subgroup $L/C_G(A)$ of finite index such that $A(\omega RG)$ is finite, then $G$ has boundedly finite orbits on $A$.

(iii) If there is an integer $b$ such that $R/b!$ is finite and $b = \text{lo}_A(G)$, then $G/C_G(A)$ includes a normal abelian subgroup $L/C_G(A)$ of finite index and finite exponent such that $A(\omega RG)$ is finite.

Next we give some specific examples of rings satisfying the conditions of Theorem 2.2. Of course, one particular interesting example is the ring of integers.

2.3. **COROLLARY** (M.R. Dixon, L.A. Kurdachenko, J. Otal [11]). Let $G$ be a group acting on the $\mathbb{Z}G$–module $A$. Then $G$ has boundedly finite orbits on $A$ if and only if $G$ includes a normal subgroup $L$ such that $G/L$ and $A(\omega \mathbb{Z}L)$ are finite.

Next result generalizes this. An infinite Dedekind domain $D$ is said to be a Dedekind $\mathbb{Z}_0$–domain if for every maximal ideal $P$ of $D$, the quotient ring $D/P$ is finite (see [9, Chapter 6], for example). If $F$ is a finite field extension of $\mathbb{Q}$ and $R$ is a finitely generated subring of $F$, then $R$ is an example of a Dedekind $\mathbb{Z}_0$–domain.

2.4. **COROLLARY** (M.R. Dixon, L.A. Kurdachenko, J. Otal [11]). Let $G$ be a group, $D$ be a Dedekind $\mathbb{Z}_0$–domain and $A$ be a $DG$–module. Then $G$ has boundedly finite orbits on $A$ if and only if there exists a normal abelian subgroup $L/C_G(A)$ of $G/C_G(A)$ of finite index and finite exponent such that $A(\omega DG)$ is finite.
For the case when a ring of scalars is a field, we obtain

2.5. THEOREM (M.R. Dixon, L.A. Kurdachenko, J. Otal [11]). Let $G$ be a group, $F$ be a field of characteristic $p > 0$ and $A$ an $FG$–module. Suppose that $G$ acts on $A$ with boundedly finite $G$–orbits. Then

(i) $G/C_G(A)$ includes a normal abelian $p$–subgroup $L/C_G(A)$ of finite exponent such that $G/L$ is finite.

(ii) $A$ includes an $FG$–submodule $C$ such that $\dim_F(C)$ is finite and $L$ acts trivially on $C$ and $A/C$.

The next result considers the situation when $G/C_G(A)$ is finite.

2.6. THEOREM (M.R. Dixon, L.A. Kurdachenko, J. Otal [11]). Let $G$ be a group, $F$ be a field and $A$ an $FG$–module. Suppose that $G$ acts on $A$ with boundedly finite $G$–orbits. Assume that if $\text{char} F = p > 0$, then $G/C_G(A)$ is a $p'$–group. Then $G/C_G(A)$ is finite.

In particular, if $F$ is a field of characteristic 0, then $G$ acts on the $FG$–module $A$ with boundedly finite $G$–orbits if and only if $G/C_G(A)$ is finite.

We consider now the following generalization. If a group $G$ acts on $A$ with finite $G$–orbits, then an $FG$–submodule $aFG$ has finite dimension over $F$.

Let $F$ be a field, $A$ be a vector space over $F$ and $G$ a subgroup of $\text{GL}(F, A)$. We say that $G$ is a linear group with finite dimensional $G$–orbits (or that $A$ has finite dimensional $G$–orbits) if the $G$–orbit $aG$ generated a finite dimensional subspace for each element $a \in A$.

As we have seen above, if a group $G$ has finite $G$–orbits then $G$ has finite dimensional $G$–orbits, but the converse is false. Every ordinary finite dimensional linear group $G$ is a group with finite dimensional $G$–orbits. But we have seen above that if a finite dimensional linear group $G$ has finite $G$–orbits, then $G$ is finite.

We say that a linear group $G$ has boundedly finite dimensional orbits on $A$ if there is a positive integer $b$ such that $\dim_F(aFG) \leq b$ for each element $a \in A$. Put

$$\text{md}(G) = \max \{ \dim_F(aFG) \mid a \in A \}.$$ 

Every linear group $G$ defined over a finite dimensional vector space $A$ is a group with boundedly finite dimensional orbits.

In view of the Neumann's result, a natural question arises: when is $\dim_F(A(oFG))$ finite? An easy computation shows that $aFG \leq A(oFG) + aF$ for each $a \in A$, and hence if $\dim_F(A(oFG))$ at most $d$ then $aFG$ is of $F$–dimension at most $d + 1$. Thus, if $A(oFG)$ is finite dimensional, then $G$ has boundedly finite dimensional orbits. However, as we showed above, even for linear groups having boundedly finite orbits on $A$, the converse is false. It would be interesting to know the conditions posed on a group $G$ such that $A(oFG)$ be finite dimensional.

Let $B$ be a subspace of $A$, then the norm of $B$ in $G$ is the subgroup
Norm_G(B) = \bigcap_{b \in B} N_G(bF).

Observe that Norm_G(B) is the intersection of the normalizers of all F – subspaces of B, and that G = Norm_G(A) if and only if every subspace of A is G – invariant.

The following theorem provides us with a description of linear groups having boundedly finite dimensional orbits on A.

2.7. THEOREM (M.R. Dixon, L.A. Kurdachenko, J. Otal [12]). Let F be a field, A a vector space over F and G be a subgroup of GL(F, A). Suppose that G has boundedly finite dimensional orbits on A and let b = md(G). Then

(i) A has an FG – submodule D such that dim_F(D) is finite and if K = C_G(D), then K \leq Norm_G(A/D). Moreover there exist a positive integer valued function f such that dim_F(D) \leq f(b).

(ii) K is a normal subgroup of G and has a G – invariant abelian subgroup T such that A(oF(T)) \leq D and K/T is isomorphic to a subgroup of the multiplicative group of a field F.

(iii) T is an elementary abelian p – subgroup if char F = p > 0 and is a torsion – free abelian group otherwise.

In particular, G is an extension of a metabelian group by a finite dimensional linear group.

We use Theorem 2.7 to establish several properties of groups with boundedly finite dimensional orbits that are analogies to corresponding results for finite dimensional linear groups. There are numerous ways we can exploit Theorem 2.7. Here we just select some of them. It is a well – known theorem of Schur that periodic finite dimensional linear groups are locally finite.

2.8. COROLLARY (M.R. Dixon, L.A. Kurdachenko, J. Otal [12]). Suppose that G has boundedly finite dimensional orbits on A.

(i) If G is periodic then G is locally finite.

(ii) If G is locally generalized radical then G is soluble – locally finite.

(iii) If G is a periodic p′ – group, where p = char F, then the center of G includes a locally cyclic subgroup K such that G/K is soluble – by – finite.

Now we consider another topic: the reduction to the groups with finite dimensional orbits.

Let again G be a subgroup of GL(F, A). We say that G is a linear group with finite G – orbits of subspaces if the set cl_G(B) = \{ Bg \mid g \in G \} is finite for each F – subspace B of A. The groups with this property are a natural analogies of the groups with finite G – orbits of elements. Since it is clear that |cl_G(B)| = |G : N_G(B)|, it follows that G has finite G – orbits of subspaces if and only if the indexes |G : N_G(B)| are finite for all F – subspaces B of A. It is not hard to prove that if G has finite G – orbits of subspaces then dim_F(aFG) is finite, for each element a ∈ A.

Observe that if every F – subspace B is G – invariant, then G is abelian. Linear groups with finite G – orbits of subspaces can be considered as natural generalizations of abelian linear groups.
For these groups we obtain a following result.

2.9. **THEOREM** (M.R. Dixon, L.A. Kurdachenko, J. Otal [12]). Let $F$ be a field, $A$ a vector space over $F$ and $G$ be a subgroup of $\mathcal{GL}(F, A)$. Suppose that $G$ is a linear group with finite $G$–orbits of subspaces. Then a factor–group $G/\text{Norm}_G(A)$ is finite and $G$ is central – by – finite.

We say that a group has boundedly finite $G$–orbits of subspaces if there is a positive integer $b$ such that $|\text{cl}_G(B)| \leq b$ for all subspaces $B$ of $A$.

2.10. **COROLLARY** (M.R. Dixon, L.A. Kurdachenko, J. Otal [12]). Let $F$ be a field, $A$ a vector space over $F$ and $G$ be a subgroup of $\mathcal{GL}(F, A)$. Then $G$ has finite $G$–orbits of subspaces if and only if $G$ has boundedly finite $G$–orbits of subspaces.

3. **LINEAR GROUPS WITH RESTRICTION ON SUBGROUPS OF INFINITE CENTRAL DIMENSION**

If $H$ is a subgroup of $\mathcal{GL}(F, A)$, then $H$ really acts on the factor–space $A/C_A(H)$. Following to [13] we say that $H$ has finite central dimension, if $\dim_F(A/C_A(H))$ is finite. In this case $\dim_F(A/C_A(H)) = \text{centdim}_F(H)$ will be called a central dimension of the subgroup $H$.

Let $H$ has a finite central dimension, then $A/C_A(H)$ is finite dimensional. Put $C = C_G(A/C_A(H))$, then, clearly, $C$ is a normal subgroup of $H$ and $H/C$ is isomorphic to some subgroup of $\mathcal{GL}_n(F)$ where $n = \dim_F(A/C_A(H))$. Each element of $C$ acts trivially in every factor of series $0 < C_A(H) \leq A$, so that $C$ is an abelian subgroup. Moreover, if $\text{char} F = 0$, then $C$ is torsion – free; if $\text{char} F = p > 0$, then $C$ is an elementary abelian $p$ – subgroup. Hence, the structure of $H$ in general is defined by the structure of $G/C$, which is an ordinary finite dimensional linear group.

Let $G \leq \mathcal{GL}(F, A)$ and let $\mathcal{L}_{\text{id}}(G)$ be a set of all proper subgroups of $G$ having infinite central dimension. In the paper [13], it has been proved that if every proper subgroup of $G$ has finite central dimension, then either $G$ has a finite central dimension or $G$ is a Prüfer $p$ – group for some prime $p$ (under some natural restriction on $G$). This shows that it is natural to consider such linear groups $G$, in which the family $\mathcal{L}_{\text{id}}(G)$ is “very small” in some particular sense. But what means “very small” for infinite groups? One of the natural approaches possible here is employing of finiteness condition. More precisely, it is natural to consider the groups in which the family $\mathcal{L}_{\text{id}}(G)$ satisfies some strong finiteness condition. In the paper [14] we considered some of such situations. In particular, the linear groups in which the family $\mathcal{L}_{\text{id}}(G)$ satisfies minimal and maximal condition and some rank restriction were considered. The weak minimal and weak maximal condition are the natural group – theoretical generalizations of the ordinary minimal and maximal condition. These conditions have been introduced by R. Baer [15] and D.I. Zaitsev [16]. The definition of the weak minimal condition in the most general form is following.
Let $G$ be a group and $\mathcal{M}$ be a family of subgroups of $G$. We say that $\mathcal{M}$ satisfies the weak maximal (respectively minimal) condition or $G$ satisfies the weak maximal (respectively minimal) condition for $\mathcal{M}$—subgroups, if for every ascending (respectively descending) series \{ $H_n$ | $n \in \mathbb{N}$ \} subgroups of family $\mathcal{M}$ there exists a number $m \in \mathbb{N}$ such that the indexes $|H_{n+1}:H_n|$ (respectively $|H_n:H_{n+1}|$) are finite for all $n \geq m$.

The groups with the weak minimal and maximal conditions for distinct important families of subgroups have been studied by many authors (see, for example, the book [17, 5.1] and the surveys [18]).

We say that a group $G \leq \text{GL}(F, A)$ satisfies a weak minimal (respectively maximal) condition for subgroups of infinite central dimension or shortly $\text{Wmin}$—icd (respectively $\text{Wmax}$—icd), if a family $\mathcal{L}_{icd}(G)$ satisfies the weak maximal (respectively minimal) condition.

The first results about linear groups satisfying the conditions $\text{Wmin}$—icd and $\text{Wmax}$—icd have been obtained in paper [19]. More precisely, this paper was devoted to the study of such periodic groups. The main results of this paper are:

3.1. **THEOREM (J. M. Muñoz – Escolano, J. Otal and N.N. Semko [19]).** Let $F$ be a field, $A$ a vector space over $F$ and $G$ be a locally soluble periodic subgroup of $\text{GL}(F, A)$. Suppose that $G$ has infinite central dimension and satisfies $\text{Wmin}$—icd or $\text{Wmax}$—icd. The following assertions hold:

1. If $\text{char } F = 0$, then $G$ is a Chernikov group.
2. If $\text{char } F = p > 0$, then either $G$ is a Chernikov group or $G$ has a series of normal subgroups $H \leq D \leq G$ satisfying the following conditions:
   1a. $H$ is a nilpotent bounded $p$—subgroup.
   1b. $D = H \triangleleft Q$ for some non—identity divisible Chernikov subgroup $Q$ such that $p \not\in \Pi(Q)$.
   1c. $H$ has finite central dimension, $Q$ has infinite central dimension.
   1d. If $K$ is a Prüfer $q$—subgroup of $Q$ and $K$ has an infinite central dimension, then $H$ has a finite $K$—composition series.
   1e. $G/D$ is finite.

3.2. **COROLLARY (J. M. Muñoz – Escolano, J. Otal and N.N. Semko [19]).** Let $F$ be a field, $A$ a vector space over $F$ and $G$ be a locally soluble periodic subgroup of $\text{GL}(F, A)$. Then the following conditions are equivalent:

1. $G$ satisfies the weak minimal condition on subgroups of infinite central dimension;
2. $G$ satisfies the weak maximal condition on subgroups of infinite central dimension;
3. $G$ satisfies the minimal condition on subgroups of infinite central dimension.

3.3. **COROLLARY (J. M. Muñoz – Escolano, J. Otal and N.N. Semko [19]).** Let $F$ be a field, $A$ a vector space over $F$ and $G$ be a periodic locally nilpotent subgroup of $\text{GL}(F, A)$. Suppose that $G$ has infinite central dimension. Then the following conditions are equivalent:
(i) $G$ satisfies the weak minimal condition on subgroups of infinite central dimension;
(ii) $G$ satisfies the weak maximal condition on subgroups of infinite central dimension;
(iii) $G$ satisfies the minimal condition on subgroups of infinite central dimension;
(iv) $G$ is Chernikov; and
(v) $G$ satisfies the minimal condition on all subgroups.

For non-periodic groups, the situation is more complicated. The study of locally nilpotent linear groups satisfying $\text{Wmin} – \text{id}$ and $\text{Wmax} – \text{id}$ have been initiated in the papers [20, 21]. The first result shows the minimax groups are nilpotent groups with this conditions.

3.4. THEOREM (L.A. Kurdachenko, J. M. Muñoz – Escolano and J. Otal [20]). Let $F$ be a field, $A$ a vector space over $F$ and $G$ be a subgroup of $\mathcal{GL}(F, A)$ having infinite central dimension. Suppose that $H$ is a normal subgroup of $G$ such that $G/H$ is nilpotent. If $G$ satisfies either $\text{Wmin} – \text{id}$ or $\text{Wmax} – \text{id}$, then $G/H$ is minimax. In particular, if $G$ is nilpotent, then $G$ is minimax.

Further results pertain to the case of a prime characteristic.

3.5. THEOREM (L.A. Kurdachenko, J. M. Muñoz – Escolano and J. Otal [20]). Let $F$ be a field of prime characteristic, $A$ a vector space over $F$ and $G$ be a locally nilpotent subgroup of $\mathcal{GL}(F, A)$ having infinite central dimension. If $G$ satisfies either $\text{Wmin} – \text{id}$ or $\text{Wmax} – \text{id}$, then $G/\text{Tor}(G)$ is minimax. In particular, if $\text{Tor}(G)$ has infinite central dimension, then $G$ is minimax.

Here $\text{Tor}(G)$ is a maximal normal periodic subgroup of $G$. If $G$ is a locally nilpotent group, then $\text{Tor}(G)$ consists of all elements of finite order, so that $G/\text{Tor}(G)$ is torsion-free.

Let $\mathcal{F}$ be the class of finite groups. If $G$ is a group, then the intersection $G_{\mathcal{F}}$ of all subgroups of $G$, having finite index, is called the finite residual of $G$.

3.6. THEOREM (L.A. Kurdachenko, J. M. Muñoz – Escolano and J. Otal [20]). Let $F$ be a field of prime characteristic, $A$ a vector space over $F$ and $G$ be a locally nilpotent subgroup of $\mathcal{GL}(F, A)$ having infinite central dimension. If $G$ satisfies either $\text{Wmin} – \text{id}$ or $\text{Wmax} – \text{id}$, then $G/G_{\mathcal{F}}$ is minimax and nilpotent.

Let $\mathcal{N}$ be the class of nilpotent groups. The intersection $G_{\mathcal{N}}$ of all normal subgroups $H$ such that $G/H$ is nilpotent, is called the nilpotent residual of $G$.

3.7. THEOREM (L.A. Kurdachenko, J. M. Muñoz – Escolano and J. Otal [20]). Let $F$ be a field of prime characteristic, $A$ a vector space over $F$ and $G$ be a locally nilpotent subgroup of $\mathcal{GL}(F, A)$ having infinite central dimension. If $G$ satisfies either $\text{Wmin} – \text{id}$ or $\text{Wmax} – \text{id}$, then $G/G_{\mathcal{N}}$ is minimax.
For the case of non-finitary linear groups, the following results were obtained.

3.8. **THEOREM** (L.A. Kurdachenko, J. M. Muñoz – Escolano, J. Otal and N.N. Semko [21]). Let $F$ be a field, $A$ a vector space over $F$ and $G$ be a locally nilpotent subgroup of $\text{GL}(F, A)$ having infinite central dimension. If $G$ is not finitary and satisfies $\text{Wmin}$ – icd, then $G$ is minimax.

For the case of hypercentral groups and prime characteristic the study was completed. The following result shows it.

3.9. **THEOREM** (L.A. Kurdachenko, J. M. Muñoz – Escolano, J. Otal and N.N. Semko [21]). Let $F$ be a field of prime characteristic, $A$ a vector space over $F$ and $G$ be a hypercentral subgroup of $\text{GL}(F, A)$ having infinite central dimension. If $G$ satisfies $\text{Wmin}$ – icd, then $G$ is minimax.

We observe that for the condition $\text{Wmax}$ – icd the similar results is not true. In the paper [21], an example of hypercentral linear groups over the field of prime characteristic satisfying $\text{Wmax}$ – icd which is not minimax was constructed.

The paper [22] initiated the study of soluble linear groups satisfying $\text{Wmin}$ – icd. The following main result of this paper shows that their structure is rather similar to the structure of finite dimensional soluble groups.

Let $G \leq \text{GL}(F, A)$. We recall that an element $x \in G$ is called unipotent if there is a positive integer $n$ such that $A(x – 1)^n = 0$. A subgroup $H$ of $G$ is called unipotent if every element of $H$ is unipotent. A subgroup $H$ of $G$ is called boundedly unipotent if there is a positive integer $n$ such that $A(x – 1)^n = 0$ for each element $x \in H$.

3.10. **THEOREM** (L.A. Kurdachenko, J. M. Muñoz – Escolano, J. Otal [22]). Let $F$ be a field, $A$ a vector space over $F$ and $G$ be a soluble subgroup of $\text{GL}(F, A)$. Suppose that $G$ has infinite central dimension and satisfies $\text{Wmin}$ – icd. If $G$ is not minimax, then $G$ satisfies the following conditions:

(i) $G$ has a normal boundedly unipotent subgroup $L$ such that $G/L$ is minimax;
(ii) $L$ has finite central dimension;
(iii) if $\text{char}(F) = 0$, then $L$ is nilpotent torsion – free subgroup;
(iv) if $\text{char}(F) = p$ for some prime $p$, then $L$ is nilpotent bounded $p$ – subgroup;
(v) $G$ is a finitary linear group.

If $G$ is a subgroup of $\text{GL}(F, A)$, then $G$ acts trivially on the factor – space $A/A(\text{oFG})$. Hence $G$ properly acts on the subspace $A(\text{oFG})$. As in paper [23], we define the augmentation dimension of $G$ to be the $F$– dimension of $A(\text{oFG})$ and denote it by $\text{augdim}_F(G)$. This concept is opposite in some sense to the concept of central dimension. As for the groups having finite central dimension, a group $G$ of finite augmentation dimension includes a normal abelian subgroup $C$ such that $G/C$ is an ordinary finite dimensional group. Moreover, if $\text{char} F = 0$, then $C$ is torsion – free, if $\text{char} F = p > 0$, then $C$ is an elementary abelian $p$ – subgroup. In the paper [23] the linear groups in which the set of all subgroups having infinite augmentation dimension satisfies the minimal condition have been considered. In the paper [24] the linear groups in
which the set of all subgroups having infinite augmentation dimension satisfies some rank restrictions have been considered.

We can define finitary linear groups as the groups, whose cyclic (and therefore finitely generated) subgroups have finite augmentation dimension. Therefore the following groups are the antipodes to finitary linear groups.

We say that a group \( G \leq \text{GL}(F, A) \) is called an antifinitary linear group if each proper infinitely generated subgroup of \( G \) has finite augmentation dimension (a subgroup \( H \) of an arbitrary group \( G \) is called \textit{ininitely generated} if \( H \) cannot have a finite set of generators). These groups have been studied in the paper [25]. This study splits into two cases depending on whether or not the group is finitely generated.

Let \( G \leq \text{GL}(F, A) \). Then the set

\[
\text{FD}(G) = \{ x \in G \mid \langle x \rangle \text{ has finite augmentation dimension} \}
\]

is a normal subgroup of \( G \).

Let \( D \) be a divisible abelian group and \( G \) be a subgroup of \( \text{Aut}(D) \). Then \( D \) is said to be \( G \)-\textit{divisibly irreducible} if \( D \) has no proper divisible \( G \)-invariant subgroups.

3.11. THEOREM (L.A. Kurdachenko, J. M. Muñoz – Escolano, J. Otal [25]). Let \( F \) be a field, \( A \) a vector space over \( F \) and \( G \) be a infinitely generated locally generalized radical subgroup of \( \text{GL}(F, A) \). Suppose that \( G \) is not finitary and has infinite augmentation dimension. Then \( G \) satisfies the following conditions:

(1) If the factor-group \( G/\text{FD}(G) \) is infinitely generated, then \( G \) is a Prüfer \( p \)-group for some prime \( p \).

(2) If \( G/\text{FD}(G) \) is finitely generated, then \( G \) satisfies the following conditions:

(2a) \( G = K < g > \) where \( K \) is a divisible abelian Chernikov subgroup and \( g \) is a \( p \)-element, where \( p \) is a prime such that \( p = \mid G/\text{FD}(G) \mid \);

(2b) \( K \) is a normal subgroup of \( G \);

(2c) \( K \) is \( G \)-divisibly irreducible;

(2d) \( K \) is a \( q \)-subgroup for some prime \( q \);

(2f) if \( q = p \), then \( K \) has finite special rank equal to \( p^{m-1}(p-1) \) where \( p^m = \langle g >/C < g > (K) \rangle \); and

(2g) if \( q \neq p \), then \( K \) has finite special rank \( o(q, p^m) \) where as above \( p^m = \langle g >/C < g > (K) \rangle \) and \( o(q, p^m) \) is the order of \( q \) modulo \( p^m \).

3.12. THEOREM (L.A. Kurdachenko, J. M. Muñoz – Escolano, J. Otal [25]). Let \( F \) be a field, \( A \) a vector space over \( F \) and \( G \) be a finitely generated radical subgroup of \( \text{GL}(F, A) \). Suppose that \( G \) is not finitary and has infinite augmentation dimension. Then the following conditions holds:

(1) \( \text{augdim}_{\text{FD}}(G) \) is finite;

(2) \( G \) has a normal subgroup \( U \) such that \( G/U \) is polycyclic;

(3) there is a positive integer \( m \) such that \( A(x-1)^m = <0> \) for each \( x \in U \); in particular, \( U \) is nilpotent;

(4) \( U \) is torsion-free if \( \text{char } F = 0 \) and is a bounded \( p \)-subgroup if \( \text{char } F = p > 0 \); and

(5) if
< 0 > = Z_0 ≤ Z_1 ≤ . . . Z_m = U

is an upper central series of U, then Z_1/Z_0, . . . , Z_m/Z_{m−1} are finitely generated Z< g > –modules for each element g ∈ G \ FD(G). In particular, U satisfies the maximal condition on < g > – invariant subgroups for each element g ∈ G \ FD(G).

REFERENCES


