Positive laws on large sets of generators in residually-$p$ groups

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Groups satisfying a positive law
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A reduced group word $w$ in symbols $x_1, \ldots, x_m$ is **positive** if it does not involve any inverses of the $x_i$.

Let $T$ be a subset of a group $G$. If $\alpha, \beta$ are two different positive words, $T$ satisfies the **positive law** $\alpha \equiv \beta$ if every substitution $x_i \mapsto t_i$ with $t_i \in T$ gives the same value for $\alpha$ and $\beta$. The **degree** of the law is the maximum of the lengths of $\alpha$ and $\beta$. 
Examples of groups satisfying a positive law

A group of finite exponent $e$ satisfies the exponent law $x^e \equiv 1$.

An abelian group satisfies the non-positive law $x^{-1}y^{-1}xy \equiv 1$, but also the positive law $xy \equiv yx$.

If $G/Z(G)$ satisfies the positive law $\alpha \equiv \beta$ then $G$ satisfies the positive law $\alpha\beta \equiv \beta\alpha$.

Thus all nilpotent groups of class $c$ satisfy the same positive law in two variables (the so-called Malcev law $M_c(x,y)$).

For example, $M_1(x,y)$ is $xy \equiv yx$, $M_2(x,y)$ is $xyyx \equiv yxxy$, and $M_3(x,y)$ is $xyyxyxxy \equiv yxxyxyyx$.

If $N$ satisfies a positive law and $G/N$ satisfies an exponent law then also $G$ satisfies a positive law. (Just compose the laws.)

Thus a nilpotent-by-(finite exponent) group satisfies a positive law of the form $M_c(x^e,y^e)$.
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Positive laws on generators

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Does a positive law imply nilpotent-by-(finite exponent)?

Negative general answer


But positive answer for many groups!

Burns and Medvedev; Bajorska and Macedońska (2003): For the class of locally graded groups (i.e. groups in which every non-trivial finitely generated subgroup has a proper subgroup of finite index).

Theorem

If $G$ is locally graded and satisfies a positive law of degree $n$, then:

- $G$ is nilpotent-by-(locally finite of finite exponent), and
- both the nilpotency class and the exponent are $n$-bounded (i.e. bounded in terms of $n$).
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1. Groups satisfying a positive law

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3. The case of $p$-adic analytic pro-$p$ groups
Theorem (Riley and Shumyatsky, 2005)

Let $G$ be a finitely generated residually-$p$ group. Suppose all products of the form $tu^k$, with $t$, $u$ commutators and $k \geq 0$, satisfy a positive law. Then also $G'$ satisfies a positive law.

The same result holds for simple commutators $[x_1, \ldots, x_m]$ of length $m$ and the subgroup $\gamma_m(G)$.

Question

If $G$ is finitely generated residually-$p$, does a positive law on commutators imply a positive law on the whole of $G'$?

Note that the set of commutators is a 'large' set of generators of $G'$: it is a normal subset and commutator-closed (i.e. closed under taking commutators of its elements).
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Why finitely generated residually-$p$?

Residually-$p$ groups allow a twofold analysis:

Via finite $p$-groups, by approximating the group with its finite quotients.

Via pro-$p$ groups, by embedding the group in its pro-$p$ completion.

This makes it possible to use very powerful tools.

The reason for requiring finite generation is to control the length of the elements of $G'$ as words in commutators.

Theorem

Let $G$ be a finite $p$-group generated by $x_1,\ldots,x_d$. Then every element of $G'$ can be written in the form $[g_1,x_1]\ldots[g_d,x_d]$ as a product of $d$ commutators.

That is, the commutator word $[x,y]$ has width at most $d$ in a $d$-generator finite $p$-group.
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Towards a general setting

Inspired by the problem of the commutators and the subgroup $G'$, we pose the following general question.

**Question**

Let $G$ be a finitely generated residually-$p$ group, and assume that $G$ can be generated by a subset $T$ such that:

- $T$ is a normal subset of $G$ and commutator-closed.
- $T$ satisfies a positive law.

Does it follow that the whole of $G$ satisfies a (possibly different) positive law?
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Theorem A (F-A and Shumyatsky, 2007)

Let $G$ be a $d$-generator residually-$p$ group which satisfies some law $v \equiv 1$. Suppose that $G$ has a set of generators $T$ satisfying a positive law of degree $n$, and that $T$ is a commutator-closed normal subset of $G$. Then:

There exists a finite set $P(n)$ of primes such that, if $p \notin P(n)$, then $G$ is nilpotent of bounded class.

If $T$ is power-closed then $G$ contains a normal nilpotent subgroup of bounded class and index.

Thus in both cases $G$ satisfies a positive law of bounded length.

From now onwards, we use 'bounded' to mean 'bounded in terms of the parameters involved in the problem'. Thus, in the previous theorem, bounded in terms of $d$, $p$, $v$ and $n$. 

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About the set $P(n)$

In Theorem A, $P(n)$ is a finite set of 'bad' primes for which we cannot conclude nilpotence (or even nilpotence-by-finiteness). It depends only on the degree $n$ of the positive law. The set $P(n)$ is a real obstruction: for every prime $p$, there exists a $2$-generator metabelian residually-$p$ group which is not nilpotent-by-finite but can be generated by a commutator-closed normal subset satisfying a positive law.
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Theorem

Let $G$ be a $d$-generator residually-$p$ group, and assume that one of the following two cases holds:

1. The simple commutators of length $m$ satisfy a positive law of degree $n$ and $p \not\in P(n)$.
2. All powers of simple commutators of length $m$ satisfy a positive law of degree $n$.

Then the whole of $\gamma_m(G)$ satisfies a positive law of bounded degree.

It is not clear whether the set $P(n)$ is a real obstruction in the previous theorem.
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Application to lower central subgroups

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Let $G$ be a finitely generated residually-$p$ group. If $w$ is a word, write $G^w$ for the set of all values of $w$ in $G$ and $w(G) = \langle G^w \rangle$ for the corresponding verbal subgroup. Does a positive law on $G^w$ imply a positive law on the whole of $w(G)$?

**Definition**

Let $w$ be a group word. We say that:

- $w$ is commutator-closed if the set $G^w$ of values of $w$ is commutator-closed for every group $G$.
- $w$ has finite width in a group $G$ if every element of $w(G)$ can be written as a product of at most $k$ elements of $G^w$ or $G^{-1}w$ for some fixed $k$. 

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Application to verbal subgroups

**Question**

Let $G$ be a finitely generated residually-$p$ group. If $w$ is a word, write $G_w$ for the set of all values of $w$ in $G$ and $w(G) = \langle G_w \rangle$ for the corresponding verbal subgroup. Does a positive law on $G_w$ imply a positive law on the whole of $w(G)$?
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Definition

Let $w$ be a group word. We say that:

- $w$ is commutator-closed if the set $G_w$ of values of $w$ in $G$ is commutator-closed for every group $G$.
- $w$ has finite width in a group $G$ if every element of $w(G)$ can be written as a product of at most $k$ elements of $G_w$ or $G^{-1}w$ for some fixed $k$. 
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Let $G$ be a $d$-generator residually-$p$ group, and let $w$ be a word which is commutator-closed and has finite width $m$ in $G$ (or simply bounded width in the quotients of $G$ which are finite $p$-groups). Assume that one of the following conditions holds:

1. All $w$-values in $G$ satisfy a positive law of degree $n$, and $p \not\in P(n)$.
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Then $w(G)$ satisfies a positive law of bounded degree.

Theorem B generalizes the result for simple commutators. By a result of Jaikin-Zapirain (2008), this applies to every word $w \not\in F''(F')p$ (Milnor word) which is commutator-closed: if $P$ is a $d$-generator finite $p$-group, then $w$ has $\{p,d,w\}$-bounded with in $P$. 

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1 Groups satisfying a positive law

2 Positive laws on ‘large’ sets of generators

3 The case of $p$-adic analytic pro-$p$ groups
Recall that a pro-$p$ group $G$ is $p$-adic analytic if and only if it has finite rank: for some $r$, every closed subgroup of $G$ can be topologically generated by $r$ elements.

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Let $G$ be a $p$-adic analytic pro-group. Then all words have finite width in $G$.

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It is an important open problem whether the last theorem can be made quantitative: is the width of a word $w$ bounded in terms of $p$, $w$ and the rank $r$?
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Let $G$ be a $p$-adic analytic pro-$p$ group and let $w$ be any commutator-closed word. Then in either of the following cases the verbal subgroup $w(G)$ satisfies a positive law:

- $G$ satisfies a positive law of degree $n$ and $p \not\in P(n)$.
- All powers of elements of $G$ satisfy a positive law.

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All simple commutators and all derived words are outer commutators. More generally, the polynilpotent words (i.e. compositions of simple commutators) are also outer commutator words.

**Theorem C (F-A and Shumyatsky, 2010)**

Let \( w \) be an outer commutator word which is also commutator-closed (e.g. a polynilpotent word), and let \( G \) be a \( p \)-adic analytic pro-

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Eliminating the commutator-closed condition

Theorem A applies to all residually-$p$ groups; if we restrict to $p$-adic analytic groups, we can get rid of the condition that $T$ should be commutator-closed.

Theorem D (Acciarri and F-A, 2010)

Let $G$ be a $p$-adic analytic pro-$p$ group which satisfies some law $v \equiv 1$. Suppose that $G$ can be generated topologically by a normal subset $T$ satisfying a positive law of degree $n$. If $p \not\in \mathbb{P}(n)$, then $G$ is nilpotent of bounded class.
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Powerful pro-$p$ groups without ‘bad’ primes

Recall that a pro-$p$ group is $p$-adic analytic if and only if it contains a powerful subgroup $H$ of finite index:

$$H' \leq H_p$$ if $p > 2$, or

$$H' \leq H_4$$ if $p = 2$.

The example showing that we cannot get rid of $P(n)$ in Theorem A can be chosen to be a powerful pro-$p$ group. But the generating subset $T$ has infinite width.

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Can Theorem G be improved to a quantitative version, so that the
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Let \( G \) be a \( p \)-adic analytic pro-
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