On polynomial growth joint work with O.Macedońska and W.Tomaszewski

Beata Bajorska

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Groups and Their Actions, Będlewo 2010

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What this talk is about?

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We shall characterize/describe groups of polynomial growth

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What this talk is not about?

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polynomial growth

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polynomial growth ↑ (J.Milnor, J.A.Wolf, 1968) virtual nilpotence

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polynomial growth ↓ (M.Gromov, 1981) virtual nilpotence

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The class of groups of polynomial growth

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The class of groups of polynomial growth and the class of finitely generated virtually nilpotent groups

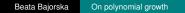
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The class of groups of polynomial growth and the class of finitely generated virtually nilpotent groups coincide.

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To be honest:



To be honest: We shall formulate a necessary and sufficient condition for a group to be virtually nilpotent

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 A group is called virtually nilpotent if it has a normal nilpotent subgroup of finite index

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- A group *G* is called **locally graded** if every nontrivial fin.gen. subgroup of *G* has a nontrivial finite image.

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- A group is called virtually nilpotent if it has a normal nilpotent subgroup of finite index
- A group *G* is called **locally graded** if every nontrivial fin.gen. subgroup of *G* has a nontrivial finite image.
- A law *u* ≡ *w* is called positive if words *u*, *w* can be written without negative powers of variables.

• Imp • • model

• *varG* - the variety generated by a group G (i.e. the smallest variety containing *G*)

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- F₂(varG) a free 2-generator group in varG (i.e. a relatively free group)

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- F₂(varG) a free 2-generator group in varG (i.e. a relatively free group)
- $F'_2(varG)$ a commutator subgroup of $F_2(varG)$
- *R*(*G*) a finite residual of *G* (i.e. the intersection of all subgroups of finite index in *G*)

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 If G is virtually nilpotent, then G' is finitely generated but not conversely (e.g. groups by S.V.Aloshyn; E.S.Golod; R.I.Grigorchuk; N.Gupta, S.Sidki; V.Sushchanskyy)

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- Relatively free groups are nice when consider the laws

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- Relatively free groups are nice when consider the laws (virtual nilpotence ⇒ positive law)

Remark None of the conditions can be omitted.

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• We obviously cannot omit (*ii*).

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- We obviously cannot omit (*ii*).
- A positive law implies (*ii*), so we cannot omit (*i*) (A.Yu.Ol'shanskii, A.Storozhev, 1996)

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Lemma If *G* is a finitely generated virtually nilpotent group then *(i)* The group *G* is locally graded

(*ii*) The commutator subgroup $F'_2(varG)$ is finitely generated.

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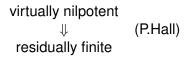
Proof of (i)

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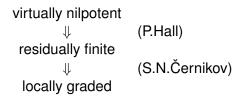
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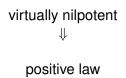
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(A.I.Mal'cev) (B.H.Neuman,T.Taylor)

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\downarrow \qquad (A.I.Mal'cev)

(B.H.Neuman,T.Taylor)

positive law

\downarrow \qquad (S.Rosset)

Milnor Property:

\forall q, h \in G \ \langle h^{-i}q h^i, \ i \in \mathbb{N} \rangle is fin.gen.
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                                                   (B.M.T. Lemma)
            F'_{2}(varG) is fin.gen.
```

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Proof of sufficiency

Lemma Let G be a finitely generated group. If

(*i*) The group *G* is locally graded (*ii*) The commutator subgroup $F'_2(varG)$ is finitely generated then *G* is virtually nilpotent.

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Proof of sufficiency

Lemma Let G be a finitely generated group. If

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Sketch of proof

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Sketch of proof

• Using (*ii*) we show that G/R(G) is virtually nilpotent

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Sketch of proof

- Using (*ii*) we show that G/R(G) is virtually nilpotent
- 2 Using 1 we show that R(G) is finitely generated

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(*i*) The group *G* is locally graded (*ii*) The commutator subgroup $F'_2(varG)$ is finitely generated then *G* is virtually nilpotent.

Sketch of proof

- Using (*ii*) we show that G/R(G) is virtually nilpotent
- 2 Using 1 we show that R(G) is finitely generated
- Solution Using and (i) we show that $R(G) = \{1\}$

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(*i*) The group *G* is locally graded (*ii*) The commutator subgroup $F'_2(varG)$ is finitely generated then *G* is virtually nilpotent.

Sketch of proof

- Using (*ii*) we show that G/R(G) is virtually nilpotent
- 2 Using 1 we show that R(G) is finitely generated
- Solution Using and (i) we show that $R(G) = \{1\}$
- Using ③ we obtain the result from ①

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Theorem A group *G* is of polynomial growth if and only if (*i*) The group *G* is locally graded (*ii*) The commutator subgroup *F*[']₂(*var G*) is finitely generated

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Thank you for your attention

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