On polynomial growth

joint work with O.Macedońska and W.Tomaszewski

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Groups and Their Actions, Będlewo 2010

What this talk is about?

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We shall characterize/describe groups of polynomial growth

What this talk is not about?

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polynomial growth

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polynomial growth
$\Uparrow$
virtual nilpotence

(J.Milnor, J.A.Wolf, 1968)

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$$\begin{array}{c} \text{polynomial growth} \\ \Downarrow \\ \text{virtual nilpotence} \end{array} \qquad \text{(M.Gromov, 1981)}$$

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The class of groups of polynomial growth

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The class of groups of polynomial growth
and
the class of finitely generated virtually nilpotent groups

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The class of groups of polynomial growth
and
the class of finitely generated virtually nilpotent groups
coincide.

To be honest:

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We shall formulate a necessary and sufficient condition for a group to be virtually nilpotent

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- A group G is called **locally graded** if every nontrivial fin.gen. subgroup of G has a nontrivial finite image.
- A **law** $u \equiv w$ is called **positive** if words u, w can be written without negative powers of variables.

Notation

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- $F_2'(varG)$ - a commutator subgroup of $F_2(varG)$
- $R(G)$ - a finite residual of G (i.e. the intersection of all subgroups of finite index in G)

Motivation

- If G is virtually nilpotent, then G' is finitely generated **but not conversely** (e.g. groups by S.V.Aloshyn; E.S.Golod; R.I.Grigorchuk; N.Gupta, S.Sidki; V.Sushchanskyy)

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- Relatively free groups are nice when consider the laws (virtual nilpotence $\Rightarrow$ positive law)

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(i) The group G is locally graded

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- We obviously cannot omit (ii).
- A positive law implies (ii), so we cannot omit (i) (A.Yu.Ol'shanskii, A.Storozhev, 1996)

Proof of necessity

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$\forall\, g,h \in G\ \ \langle h^{-i} g\, h^{i},\ i \in \mathbb{N} \rangle$ is fin.gen.

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$\Downarrow$	(B.M.T. Lemma)
$F_2'(varG)$ is fin.gen.	

Proof of sufficiency

Lemma Let G be a finitely generated group. If

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3. Using 2 and (i) we show that $R(G) = \{1\}$

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2. Using 1 we show that $R(G)$ is finitely generated
3. Using 2 and (i) we show that $R(G) = \{1\}$
4. Using 3 we obtain the result from 1

Theorem in polynomial growth version

Theorem A group G is of polynomial growth if and only if

(i) The group G is locally graded

(ii) The commutator subgroup $F_2'(var\ G)$ is finitely generated

Thank you for your attention