On commensurability of Baumslag-Solitar groups

Alexander Zakharov

Chebyshev Laboratory, St Petersburg State University

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Joint work with Montse Casals-Ruiz and Ilya Kazachkov

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- $BS(2,4)$ has infinitely generated automorphism group.

Quasi-isometry

Definition (Quasi-isometry)

A map $f : (M_1, d_1) \to (M_2, d_2)$ between two metric spaces is a quasi-isometry (qi) if there exist constants $A > 0, B \geq 0, C \geq 0$ such that

$$\frac{1}{A} d_1(x, y) - B \leq d_2(f(x), f(y)) \leq A d_1(x, y) + B,$$

$$\text{and} \quad d_2(z, f(M_1)) \leq C,$$

for all $x, y \in M_1$, $z \in M_2$.

Two f.g. groups G_1, G_2 are called quasi-isometric, denoted by $G_1 \sim_{qi} G_2$, if the Cayley graphs $Cayley(G_1)$ and $Cayley(G_2)$ are quasi-isometric.

If H is a finite index subgroup in a f.g. group G, then H and G are quasi-isometric.

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- Many groups are quasi-isometric because they are commensurable, so commensurability is the main algebraic reason for groups to be quasi-isometric. E.g.: non-abelian f.g. free groups, solvable BS groups.
- But this is not always the case: e.g., some right-angled Artin groups and non-solvable Baumslag-Solitar groups.

Solvable BS groups: qi vs commensurability

- $BS(1, n^k)$ is a finite index subgroup of $BS(1, n)$: if $BS(1, n) = \langle t, a \mid t^{-1}at = a^n \rangle$, and $H = \langle t^k, a \rangle$, then $H \cong BS(1, n^k)$.

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- It turns out these are the only ways solvable BS groups can be commensurable, and even quasi-isometric.

Theorem (Farb-Mosher, 1996)

Let $m, n \geq 1$. Then $BS(1, m) \sim_{qi} BS(1, n)$ iff $BS(1, m) \sim_c BS(1, n)$ iff there exist positive integers r, j, k such that $m = r^j$ and $n = r^k$.

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In 1998 Farb and Mosher also showed that every f.g. G qi to $BS(1, n)$ has a finite normal subgroup H such that G/H is commensurable to $BS(1, n)$.
So solvable BS groups are very "qi rigid".

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- Whyte gave a full qi classification of (non-solvable) BS groups:

Theorem (Whyte, 2004)

Let G be a non-solvable BS group. Then either $G = BS(\pm n, n)$, and so G is commensurable to $F_2 \times \mathbb{Z}$, or G is quasi-isometric to $BS(2,3)$.

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- But the general commensurability classification of non-solvable Baumslag-Solitar groups was open.

Main result: commensurability classification of BS groups

We give a complete commensurability classification of Baumslag-Solitar groups:

Theorem (Montse Casals-Ruiz, Ilya Kazachkov, A.Z., 2019)

Let $G_1 = BS(m_1, n_1)$ *and* $G_2 = BS(m_2, n_2)$, *where* $1 \leq |m_1| \leq n_1$, $1 \leq |m_2| \leq n_2$.

Then G_1 *and* G_2 *are commensurable if and only if one of the following holds:*

1. $|m_1| = |m_2| = 1$ *and* n_1, n_2 *are powers of the same integer, i.e.*

$$BS(1, n^{k_1}) \sim_c BS(1, n^{k_2}), n, k_i \in \mathbb{N};$$

2. $n_1 = n_2$ *and* $m_1 = \pm m_2$, *i.e.* $BS(m_1, n_1) \sim_c BS(\pm m_1, n_1)$;
3. $|m_1| > 1$, $|m_2| > 1$, $m_1 \mid n_1$, $m_2 \mid n_2$ *and* $\frac{n_1}{|m_1|} = \frac{n_2}{|m_2|}$, *i.e.*

$$BS(\pm k, kn) \sim_c BS(\pm l, ln), k, l, n \in \mathbb{N}, k, l > 1.$$

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- Geometric interpetation comes through *Bass-Serre theory*: such groups act on trees without edge inversions, with vertex (edge) stabilizers conjugate to vertex (edge) groups. And vice versa, group action on a tree gives rise to its decomposition as a fundamental group of a graph of groups.

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- Every finite index subgroup of a GBS group is a GBS group (by restriction of the Bass-Serre action). In particular, finite index subgroups of BS groups are GBS groups.

GBS groups: isomorphism problem

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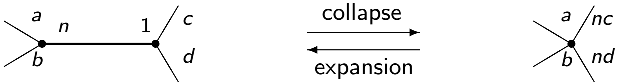
- In order to prove our main result, we prove decidability of the isomorphism problem for GBS groups in one new case.
- One of our key tools is the theory of deformation spaces, due to Clay-Forester and Guirardel-Levitt. It works for general graphs of groups, but we use it for GBS groups only.

Deformation spaces for GBS groups

Elementary deformations:



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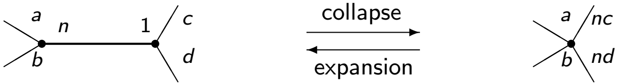

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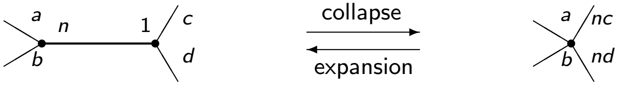

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Theorem (Forester, 2002)

Let G be a GBS group different from $BS(1,1)$ and $BS(-1,1)$. Then there is a single deformation space for G. I.e., if Γ_1 and Γ_2 are GBS graphs both defining G, then Γ_2 can be obtained from Γ_1 by some sequence of expansion and collapse moves.

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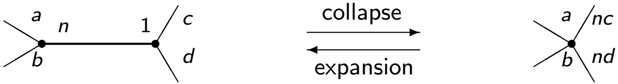

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Theorem (Forester, 2002)

Let G be a GBS group different from $BS(1,1)$ and $BS(-1,1)$. Then there is a single deformation space for G. I.e., if Γ_1 and Γ_2 are GBS graphs both defining G, then Γ_2 can be obtained from Γ_1 by some sequence of expansion and collapse moves.

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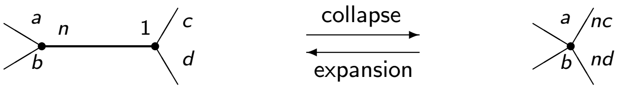

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Unfortunately, Forester's theorem is by far not sufficient to decide isomorphism of GBS groups. But we use an improved version of it due to Clay and Forester (2006), with better behaved moves.

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This provides a nice description of finite index subgroups in $BS(m, n)$ when $gcd(m, n) = 1$, but apriori not otherwise.

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- There are two cases: when $p, q > 1$ ("non-ascending" case, easier) and when $p = 1, q > 1$ ("ascending" case, harder).
- Due to modular homomorphism argument (in the next slide), if two such groups are commensurable they should be both "ascending" or both "non-ascending".

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- An element of a GBS group G is *elliptic* if it is conjugate to an element of one of the vertex groups.
- We define the *modular homomorphism* $\Delta_G : G \to \mathbb{Q}^*$ as follows. For each $g \in G$, take any elliptic $a \in G$, then we have $ga^q g^{-1} = a^p$ for some non-zero integers p, q. Let $\Delta(g) = p/q$. It's not hard to show it is well-defined.

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- $M_G = Im(\Delta_G) \subseteq \mathbb{Q}^*$ can be seen from any GBS graph representing G as the set of products of the proportions of edge labels in all closed paths.
- For $BS(m, n)$ M_G is the subgroup of $\mathbb{Q}^*$ generated by m/n.
- If H is a finite index subgroup of G, then M_H is a finite index subgroup of M_G.
- It follows that *if* $BS(m_1, n_1) \sim_c BS(m_2, n_2)$*, then* n_1/m_1 *and* n_2/m_2 *have common powers*.

Sketch of proof: non-ascending case

Lemma

Suppose that $n_1 > m_1 > 1$ *and* $n_2 > m_2 > 1$, $m_1 \nmid n_1$, $m_2 \nmid n_2$, *and the pairs* (m_1, n_1) *and* (m_2, n_2) *are distinct. Then the groups* $BS(m_1, n_1)$ *and* $BS(m_2, n_2)$ *are not commensurable.*

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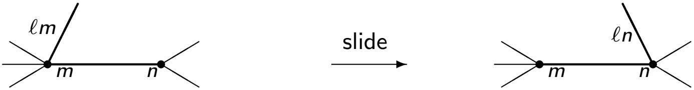

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- Let $n_1 = m_1 d_1$, $n_2 = m_2 d_2$, then by modular homomorphism argument we can suppose that d_1 and d_2 are powers of the same number: $d_1 = r^{l_1}$, $d_2 = r^{l_2}$.

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- We show that $G^{m_1}_{1,r^{l_1}}$, $G^{m_2}_{1,r^{l_2}}$ are not commensurable.

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- We solve explicitly the isomorphism problem for GBS groups which are finite index subgroups of the groups $G^{d}_{1,q}$.
- This is done by providing an appropriate "normal form" GBS graph for such subgroups. We use Clay and Forester results for that. It's tricky since two GBS graphs defining isomorphic GBS groups don't have to be related by slide moves in this case: one has to use 2 more types of trickier moves.

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3. Isomorphism and commensurability for other similar graphs of groups: e.g., instead of $\mathbb{Z}$'s we have $\mathbb{Z}^2$, etc.

Thank you!