Finite 2-nilpotent groups acting on compact manifolds

Groups and Their Actions International Conference

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Jordan groups

Definition

An infinite group $G$ is **Jordan** if there exists a positive integer $J_G$, such that every finite subgroup $K$ of $G$ contains a normal abelian subgroup whose index in $K$ is at most $J_G$.

Example (Jordan groups, algebraic)

- $\text{GL}(\mathbb{C}, n)$ for every $n$ (Camille Jordan, 1877)
- $\text{Bir}(X)$ where $X$ is an algebraic variety for
  - $X = \mathbb{P}^2_{\mathbb{C}}$, the rank 2 Cremona group (J.-P. Serre, 2009)
  - $X$ rationally connected (e.g. $X = \mathbb{P}^n_{\mathbb{C}}$) with $J_{\text{Bir}(X)}$ depending only on $\dim X$ (Prokhorov, Shramov, 2014 + Birkar 2016)
- $G$ connected algebraic group (with $J_G$ depending only on $\dim G$) (Meng, Zhang, 2017)
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Main question and result

Step 1: Group theoretic reductions

Step 2: Topological reductions

Proof of theorem

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\textbf{Diff}(M) \text{ is Jordan for manifolds } M \text{ such that}

- \( M \) compact, \( \dim(M) \leq 3 \) (Bruno Zimmermann, 2014)
- Mundet i Riera (2010-2018):
  - \( M \) is the \( n \)-torus
  - \( M \) is \( \mathbb{R}^n \) (acyclic manifolds)
  - \( M \) in the \( n \)-sphere (integral cohomology spheres)
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Question (Ghys, < 1997)

Is $\text{Diff}(M)$ Jordan for every compact manifold $M$?

Despite the positive examples above, it turned to be false:

- $\text{Diff}(\mathbb{T}^2 \times \mathbb{S}^2)$ is not Jordan (Pyber-Csikós-E. Szabó, 2014) Idea: embed the Heisenberg groups $\begin{pmatrix} 1 & \mathbb{Z}_n & \mathbb{Z}_n \\ 0 & 1 & \mathbb{Z}_n \\ 0 & 0 & 1 \end{pmatrix} \subset \text{Diff}(\mathbb{T}^2 \times \mathbb{S}^2)$ for infinitely many $n$

- Mundet i Riera (2014): higher dimensional counterexamples $M_n$. Idea: embed $H_{2n+1}(\mathbb{Z}_p) := \begin{pmatrix} 1 & \mathbb{Z}_p & \mathbb{Z}_p \\ 0 & I_n & (\mathbb{Z}_p^n)^	op \\ 0 & 0 & 1 \end{pmatrix} \subset \text{Diff}(M_n)$ for certain infinite list of primes $p$ satisfying various properties
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Non-compact case is fully solved: exists a 4-manifold containing every finite(ly presented) group (Popov, 2013).

Compact case: $\sup\{r(G) : G \in \mathcal{F}\} < \infty$ (Mann, Su, 1963). Here $r(G) := \max\{d(H) : H \subseteq G\}$ is the rank of $G$ where $d(H)$ is the cardinality of a smallest generating set of $H$.

Affirmative answer for:

- for some Heisenberg groups (Pyber-Csikós-E. Szabó, Mundet i Riera, 2014)
- all Heisenberg groups of given dimension (DSz 2017)
- every special $p$-group of order $p^n$ (DSz, 2018)

Note that all of these groups $G$ are 2-nilpotent, i.e. $[G, G] \subseteq Z(G)$.
Main question

For which families \( \mathcal{F} \) of finite groups does there exist a compact manifold \( M \) such that \( G \subseteq \text{Diff}(M) \) for every \( G \in \mathcal{F} \)?

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Theorem (DSz, 2019)

For every $r$, there exists a compact manifold $M_r$ such that $G \subseteq \text{Diff}(M_r)$ for every finite 2-nilpotent group $G$ of rank $\leq r$.

As an immediate corollary, we answer affirmatively a question of Mundet i Riera from 2018.

Corollary

For every $n$, there exists a compact manifold $M_n$ on which every finite 2-nilpotent group $G$ of order $p^n$ acts faithfully via diffeomorphisms for every $p$. 
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Overview of the group theoretic reductions

Step 1: group theoretic reductions:

1. We reduce the 2-nilpotent group $G$ to one with cyclic centre using direct products.
2. Then further to one that is generated by 2 elements using maximal central products.
3. Finally we classify the 2-generated 2-nilpotent groups having cyclic centre.
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Reduction to 2-nilpotent groups with cyclic centre

Fix $G$ a finite 2-nilpotent group of rank $\leq r$. (Wlog it is a $p$-group.) Goal: find a manifold with a $G$-action depending only on $r$.

Lemma

There exists 2-nilpotent $H_i$ with cyclic centre such that $G \subseteq \prod_{i=1}^{r} H_i$ and $d(H_i) \leq r$.

Proof.

- $Z(G) = \prod_{i=1}^{k} C_i$, if $k > 1$ consider $\prod_{i=1}^{k} G/C_i$ and use induction to embed $G$ to $\prod_{i=1}^{K} G/N_i$.
- Wlog $K \leq r$ (E. Szabó): The socle of $G$ (product of minimal normal subgroups) is $\mathbb{Z}_p^k$ for some $k \leq r$. Every non-trivial normal subgroup intersects the socle non-trivially. We select $r$ normal subgroups $N_i \subseteq G$ from the list with trivial intersection.
- $H_i := G/N_i$, $d(H_i) \leq d(G) \leq r$. 

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Maximal central products

**Definition (Maximal central product)**

Given an isomorphism \( \varphi : D_1 \to D_2 \) for \( D_i \subseteq Z(G_i) \), set

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G_1 \gamma_{\varphi} G_2 := G_1 \times G_2 / \{(z, \varphi(z)^{-1}) : z \in D_1 \},
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the central product along \( \varphi \) amalgamating \( D_1 \) and \( D_2 \).

We call \( \varphi \) a maximal, if \( \varphi \) cannot be extended further to an isomorphism between central subgroups, and in this case the call the central product is maximal and denote it by \( \overline{G_1 \gamma_{\varphi} G_2} \).

**Lemma**

\( Z(G_1 \gamma_{\varphi} G_2) \) is cyclic if and only if \( Z(G_1) \), \( Z(G_2) \) are both cyclic and \( \varphi \) is a maximal central isomorphism.
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the *central product* along $\varphi$ amalgamating $D_1$ and $D_2$.

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Reduction to 2-generated groups of cyclic centre

Let $H$ be any factor from the previous embedding.

**Lemma (2-nilpotent group with cyclic centre)**

There exists 2-generated 2-nilpotent groups $E_1, \ldots, E_n$ all with cyclic centre where $n \leq \lceil r/2 \rceil$, such that

$$H \cong \ldots ((E_1 \bar{\varphi}_1 E_2) \bar{\varphi}_2 E_3) \bar{\varphi}_3 \ldots) \bar{\varphi}_{n-1} E_n,$$

and the isomorphisms class of $H$ is independent of $\varphi_i$.

**Proof.**

- Find minimal generating set $\{\alpha_1, \alpha_2\} \cup S$ of $E$ such that $H' = \langle [\alpha_1, \alpha_2] \rangle$, $[\alpha_i, s] = 1$ for all $s \in S$.
- For $H_0 := \langle S \rangle$ and $E_n := \langle \alpha, \beta \rangle$, we have $H \cong H_0 \bar{\varphi} E_n$ and use induction on $d$.
- Independence follows from last part of the next Lemma.
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$$H \cong \ldots (((E_1 \bar{\varphi}_1 E_2) \bar{\varphi}_2 E_3) \bar{\varphi}_3 \ldots) \bar{\varphi}_{n-1} E_n,$$

and the isomorphisms class of $H$ is independent of $\varphi_i$.

**Proof.**

- Find minimal generating set $\{\alpha_1, \alpha_2\} \cup S$ of $E$ such that $H' = \langle [\alpha_1, \alpha_2] \rangle$, $[\alpha_i, s] = 1$ for all $s \in S$.
- For $H_0 := \langle S \rangle$ and $E_n := \langle \alpha, \beta \rangle$, we have $H \cong H_0 \bar{\varphi} E_n$ and use induction on $d$.
- Independence follows from last part of the next Lemma.
Reduction to 2-generated groups of cyclic centre

Let $H$ be any factor from the previous embedding.

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Let $E$ be an element of the previous decomposition.

**Lemma (2-generated 2-nilpotent group with cyclic centre)**

1. $Z(E) \subseteq E$ gives $1 \to \mathbb{Z}_a \to E \to \mathbb{Z}_c \times \mathbb{Z}_c \to 1$ for some unique integers $c | a$,

2. $E = \langle \alpha, \beta, \gamma : \gamma = [\alpha, \beta], 1 = \gamma^c = [\alpha, \gamma] = [\beta, \gamma], \alpha^a = \gamma^{c_1}, \beta^c = \gamma^{c_2} \rangle$ for some integers $c_1, c_2 | c$. There are two types:
   - either $\langle \gamma \rangle = [E, E] = Z(E)$, $c = a$,
   - or $\langle \gamma \rangle = [E, E] \subsetneq Z(E) = \langle \alpha^c \rangle$, $c < a$, $c_1 = 1$

3. Every automorphism of a central subgroup can be extended to $E$.

**Proof.**

Pull back suitable generators of $E/E'$. 

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Overview of the topological reductions

Step 2: Topological reductions:
Reduce from \( \text{Diff}(M) \) for fixed \( M \), to \( \text{Diff}(L) \) for a (possibly varying) \( L \) where \( L \rightarrow X \) is a line bundle over a fixed \( X \).

Details:

- We define the notion of **central group action on line bundles**.
- We show that two such action induce another one such a maximal central product of groups.
- We show that every central action induce a faithful action on a compact space *depending only on the base space*. 
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Central action

**Definition**

A *central action* of a finite group $G$ on a line bundle $\pi : L \to X$ is a pair of group morphisms $(\varrho : G \to \text{Diff}(L), \overline{\varrho} : G \to \text{Diff}(X))$ such that

- **equivariant:** $\overline{\varrho}(g) \circ \pi = \pi \circ \varrho(g)$ for every $g \in G$
- **linear:** $\varrho(g) : L_x \to L_{\overline{\varrho}(g(x))}$ is $\mathbb{C}$-linear for every $g \in G$
- **central (non-standard notion):** $X$ is compact, connected, $H^{2\bullet}(X, \mathbb{Z})$ torsion free; $\varrho$ is injective; $\overline{\varrho}(g)$ is homotopic to the identity on $X$ for every $g \in G$; $\text{Stab}_{\overline{\varrho}}(x) = Z(G)$ for every $x \in X$.

$Z(G)$ is necessarily cyclic.

\[ 1 \longrightarrow Z(G) \longrightarrow G \longrightarrow G/Z(G) \longrightarrow 1 \]

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Building central actions on maximal central products

Lemma (Central product construction)

Any two central actions \( \varrho_i : G_i \circlearrowleft \pi_i \) induce a natural central of some \( G_1 \tilde{\varphi} G_2 \).

Proof.

Quotient the natural \( G_1 \times G_2 \)-action on \( \pi_1 \boxtimes \pi_2 \) by its kernel.
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Trivialising central actions

Proposition (Trivialising actions)

There exists $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that whenever $\varrho$ is a central action of $G$ on $\pi : L \to X$, then $G \subseteq \text{Diff}(X \times \mathbb{C}P^f(\dim X))$.

Remark

$X \times \mathbb{C}P^f(\dim X)$ is compact and is independent of $G$ (which is the main goal), although typically $L$ very much can depend on $G$.

Proof Strategy:

1. Find a direct complement $\pi_\perp$ of fixed rank $N$ with a compatible $G$-action using $K$-theory and by hand
2. This gives a faithful action on the fixed space $X \times \mathbb{C}^{N+1}$
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Proof.

- By assumptions and using Atiyah—Hirzebruch theorem, we have:

\[
\begin{align*}
    K^0(X) \otimes \mathbb{Q} & \xleftarrow{p^*} K^0(X/\varphi) \otimes \mathbb{Q} \\
    \downarrow \text{ch} & \quad \downarrow \text{ch} \\
    H^2\text{•}(X, \mathbb{Q}) & \xleftarrow{p^*} H^2\text{•}(X/\varphi, \mathbb{Q})
\end{align*}
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where \( p : X \to X/\varphi \) is the projection. So for some \( d \in \mathbb{N}_0 \),
\( \text{ch}^{-1}(d \cdot H^2\text{•}(X, \mathbb{Z})) \) carries a natural \( G \)-action compatible with \( \varphi \).

- Enough to find a multiset \( A \ni 1 \) of integers of fixed size such that

\[
\text{ch}(\bigoplus_{a \in A} \pi \otimes a) - |A| = \sum_{k=1}^{n} \sum_{a \in A} \frac{a^k}{k!} c_1(\pi)^{k} \in d \cdot H^2\text{•}(X, \mathbb{Z}).
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- This leads to the following Waring-type problem with \( \delta = dn! \).
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**Number theory**

**Lemma**

For arbitrary natural $n, \delta$, every initial sequence of integers $a_1, \ldots, a_m$ can be extended to $a_1, \ldots, a_m, a_{m+1}, \ldots, a_C$ of length $C = C(n, m)$ such that

$$\delta \mid \sum_{i=1}^{C} a_i^k \quad \forall 1 \leq k \leq n.$$ 

The independence of $G$ on the manifold we are looking for translates to the independence of $C(n, |M|)$ on $\delta$.

**Proof.**

- Modulo Waring problem: modulo any number, $-1$ can be expressed as a sum of at most $W_k$ $k$th powers.
- Expand $(-\sum M + \sum_{a \in M} a) \prod_{k=2}^{n} (1 + a_{k,1} + \ldots a_{k,W_k})$. $k$th power sum $\in \delta \mathbb{Z}$.
# Number theory

## Lemma

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Proof.

Let $G$ be a 2-nilpotent group of rank $\leq r$.

- By step 1: $G \subseteq \prod_{i=1}^{r} H_i$ and $H_i \cong E_{i,1} \overline{\ast} E_{i,2} \overline{\ast} \cdots \overline{\ast} E_{i,\lceil r/2 \rceil}$ and each $E_{i,j}$ is given by a concrete presentation.
- One can construct a central action of each $E_{i,j}$ over the fixed $\mathbb{T}^2$. (Idea: the Appell–Humbert theorem describes fully the holomorphic vector bundles over $\mathbb{T}^2$ and we lift the action of $\mathbb{Z}_c^2$ on $\mathbb{T}^2$.)
- Apply the Central Product Construction $\lceil r/2 \rceil - 1$ times to get a central action of each $H_i$ (over $\mathbb{T}^{2\lceil r/2 \rceil}$).
- The Trivialising Proposition gives a compact manifold $X_r$ independent of $H_i$ such that each $H_i \subseteq \text{Diff}(X_r)$.
- Then $G \subseteq \text{Diff}(M_r)$ where $M_r = (X_r)^r$ is a manifold as required. $\square$
Proof of main theorem

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- By step 1: $G \subseteq \prod_{i=1}^{r'} H_i$ and $H_i \cong E_{i,1} \hat{\times} E_{i,2} \hat{\times} \cdots \hat{\times} E_{i,[r/2]}$ and each $E_{i,j}$ is given by a concrete presentation.
- One can construct a central action of each $E_{i,j}$ over the fixed $\mathbb{T}^2$.
  (Idea: the Appell–Humbert theorem describes fully the holomorphic vector bundles over $\mathbb{T}^2$ and we lift the action of $\mathbb{Z}_2^c$ on $\mathbb{T}^2$.)
- Apply the Central Product Construction $[r/2] - 1$ times to get a central action of each $H_i$ (over $\mathbb{T}^{2,[r/2]}$).
  - The Trivialising Proposition gives a compact manifold $X_r$ independent of $H_i$ such that each $H_i \subseteq \text{Diff}(X_r)$.
  - Then $G \subseteq \text{Diff}(M_r)$ where $M_r = (X_r)^r$ is a manifold as required.
Proof of main theorem

Proof.

Let \( G \) be a 2-nilpotent group of rank \( \leq r \).

By step 1: \( G \subseteq \prod_{i=1}^{r} H_i \) and \( H_i \cong E_{i,1} \bar{\gamma} E_{i,2} \bar{\gamma} \cdots \bar{\gamma} E_{i,[r/2]} \) and each \( E_{i,j} \) is given by a concrete presentation.

One can construct a central action of each \( E_{i,j} \) over the fixed \( \mathbb{T}^2 \).

(Idea: the Appell–Humbert theorem describes fully the holomorphic vector bundles over \( \mathbb{T}^2 \) and we lift the action of \( \mathbb{Z}_c^2 \) on \( \mathbb{T}^2 \).)

Apply the Central Product Construction \( \lceil r/2 \rceil - 1 \) times to get a central action of each \( H_i \) (over \( \mathbb{T}^2 \lceil r/2 \rceil \)).

The Trivialising Proposition gives a compact manifold \( X_r \) independent of \( H_i \) such that each \( H_i \subseteq \text{Diff}(X_r) \).

Then \( G \subseteq \text{Diff}(M_r) \) where \( M_r = (X_r)^r \) is a manifold as required. \( \Box \)
Proof of main theorem

**Proof.**

- Let \( G \) be a 2-nilpotent group of rank \( \leq r \).
- By step 1: \( G \subseteq \prod_{i=1}^{r} H_i \) and \( H_i \cong E_{i,1} \times E_{i,2} \times \cdots \times E_{i,\lceil r/2 \rceil} \) and each \( E_{i,j} \) is given by a concrete presentation.
- One can construct a central action of each \( E_{i,j} \) over the fixed \( \mathbb{T}^2 \).
  (Idea: the Appell–Humbert theorem describes fully the holomorphic vector bundles over \( \mathbb{T}^2 \) and we lift the action of \( \mathbb{Z}_c^2 \) on \( \mathbb{T}^2 \).)
- Apply the Central Product Construction \( \lceil r/2 \rceil - 1 \) times to get a central action of each \( H_i \) (over \( \mathbb{T}^2 \lceil r/2 \rceil \)).
- The Trivialising Proposition gives a compact manifold \( X_r \) independent of \( H_i \) such that each \( H_i \subseteq \text{Diff}(X_r) \).
- Then \( G \subseteq \text{Diff}(M_r) \) where \( M_r = (X_r)'^r \) is a manifold as required.
Thank you for your attention!