On weak Sierpiński subsets in groups and free subgroups

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Definition 1

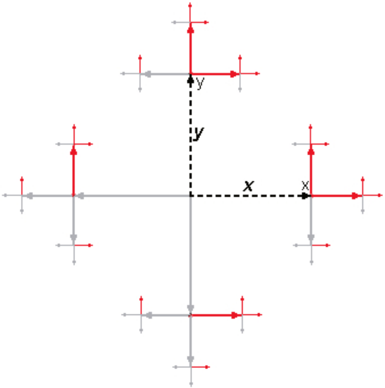
E is a Sierpiński set in metric space (or group) if for any $p \in E$, $E \cong E \setminus \{p\}$.

Definition 2

Let G - group, $E \subset G$. A weak Sierpiński subset ((σ, τ)-wS-subset) is a subset E such that for some $\sigma, \tau \in G$ and $p \neq q \in E$, we have $\sigma E = E \setminus \{p\}$ and $\tau E = E \setminus \{q\}$.

Example 1

$\mathbb{F}_2 = \langle x, y \rangle$, $E = E_x \cup E_y$, E_x, E_y - *words ending with* x, y. *Then* $xE = E \setminus \{x\}$, $yE = E \setminus \{y\}$.



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Theorem 4

$\exists E \subset \mathbb{R}^3$: *E is a Sierpiński set.*

Theorem 5

(Straus) Let $\mathbb{F}_n$ *- free group of rank* $n \geqslant 2$*,*
$\mathfrak{m}$ *- cardinal,* $|\mathbb{F}_n^{\mathfrak{m}}| = |\mathbb{F}_n|$*.*
Then

$$1) \quad \exists U \subset \mathbb{F}_n : \quad |U| = |\mathbb{F}_n|,$$

$$2) \quad \forall Q \subset U,\ |Q| \leq \mathfrak{m}\ \exists p_Q \in \mathbb{F}_n : \quad p_Q U = U \setminus Q.$$

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Corollary 1

(Straus) Let S be a sphere in $\mathbb{R}^3$. Then
$\exists U \subset S \ \forall p \in U \ \exists$ rotation ρ of S: $\rho U = U \setminus \{p\}$.

Definition 3

A set $E \subset \mathbb{R}^n$ is paradoxical if $\varphi, \psi : E \to E$ are injections which are piecewise isometries with finitely many pieces such that $\varphi(E) \cap \psi(E) = \emptyset$
If G is a group of isometries of $\mathbb{R}^n$ and $\varphi, \psi \in G$ then E is called G-paradoxical.

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Definition 4

A set $E \subset \mathbb{R}^n$ is uniformly discrete if

$$\exists \epsilon > 0 \ \forall e_1, e_2 \in E, \ e_1 \neq e_2, \quad d(e_1, e_2) \geqslant \epsilon.$$

Proposition 1

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Pruss' example:
φ - rotation around the z-axis about 1 radian,
ψ - translation $[1, 0, 0]$.
E - orbit with origin generated by φ and ψ without inversions.

$|x|$ - the Euclidean norm of x.

Theorem 6

(Mycielski, Tomkowicz 2018) Let **A** *be a boolean algebra of subsets of* $\mathbb{R}^n$*, and* **B** *be the ring of bounded sets of* **A***. Let* G *be any subgroup of the group of isometries of* $\mathbb{R}^n$ *such that* **A** *is* G*-invariant, and* $E \in$**A** *be a* G*-paradoxical set with pieces belonging to* **A***. Let* m *be a finitely additive and finite* G*-invariant measure over* **B** *such that there exists a constant* C *and for every* $r > 1$ *and every* $X \subset E$ *such that* $X \in$**A***, if* $|x| \leq r$ *for all* $x \in X$*, then*

$$m(X) \leq Cr^n.$$

Then $m(X) = 0$ *for all* $X \subset E$ *such that* $X \in$**B***.*

Theorem 7

(Mycielski, Tomkowicz 2018) Let S be a semigroup of isometries of $\mathbb{R}^n$ and let $E \subset \mathbb{R}^n$ be an uniformly discrete set. Suppose that for some point $x \in E$, $u(x) \neq v(x)$ for all $u, v \in S$, $u \neq v$. Then E contains at most one point p such that $\sigma(E) = E \setminus \{p\}$ for some $\sigma \in S$.

Sketch of the proof of Theorem 6:

Lemma 1

Let $\sigma, \tau \in G(\mathbb{R}^n)$ and E be a discrete subset of $\mathbb{R}^n$ with

$$\sigma(E) = E \setminus \{p\},\ \tau(E) = E \setminus \{q\},$$

where $p, q \in E$ and $p \neq q$.
Then the semigroup S generated by σ and τ has no fixed point in $\mathbb{R}^n$.

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$|u^t(c) - y|$ is constant for $t \in \mathbb{N}$,
$y,\ u(y),\ u^2(y),\ \cdots$ is not discrete. $\square$

Lemma 2

Let E be any set and σ and τ be injections of E into E such that

$$\sigma(E) = E \setminus \{p\},\ \tau(E) = E \setminus \{q\},$$

where $p, q \in E$ and $p \neq q$. Let S be the cancellative semigroup generated by σ, τ and p, q are not fixed points of any element of S. Then S is free, freely generated by σ, τ.

Let $u = u(\sigma, \tau), v = v(\sigma, \tau)$, let $u = v$ be cancelled and the shortest relation in S,
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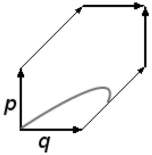

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$\sigma S \cap \tau S = \emptyset$.
Hence S is paradoxical (S acts freely on $S(x)$).
$S(x)$ is uniformly discrete as a subset of E, a contradiction with Theorem 6.

Conjecture (Mycielski, Tomkowicz):

If a group G consists of wS-subset then has nonabelian subgroup.

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Any abelian group contains no wS-subset.

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Let E be a (g, h)-wS-subset. By definition:

$$(hg)E = h(E \setminus \{a\}) = E \setminus \{b, ha\},$$

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Hence:

$$a = b \text{ or } a = ha \text{ and } h = 1.$$

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Let E be a (g,h)-wS-subset of G. Then g, h are not torsion.

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Fact 2

Let E be a (g, h)-wS-subset of G. Then g, h are not torsion.

$gE \subset E \Rightarrow g^n E \neq E$ and similarly, $h^n E \neq E$.

Theorem 8

(Bier, de Cornulier, Słanina) Let G be a group with a (g, h)-wS-subset. Then the subgroup $H = \langle g, h \rangle$ is either free over (g, h), or there exists $k \geq 2$ such that it has the presentation $H = G_k = \langle g, h \mid (h^{-1}g)^k \rangle$.

Proposition 2

(Bier, de Cornulier, Słanina) In G_k, there are exactly k subsets E such that $gE = E \smallsetminus \{1\}$ and hE is E minus a singleton; for $k-1$ of them this yields a wS-subset. More precisely, in the Schreier graph, these are the subsets E_ℓ defined by cutting along the edge $(g^{-1}, 1)$ and the edge $(g^{-1}(hg^{-1})^{\ell-1}, (hg^{-1})^\ell)$ for some $1 \leq \ell \leq k$. We have $hE_\ell = E_\ell \smallsetminus \{b_\ell\}$ with $b_\ell = (hg^{-1})^\ell$, which for $\ell = k$ equals 1 and otherwise is not 1 (so we have a wS-subset). In particular the right action of G_k on the set of (g, h)-wS-subsets is free and has exactly $k-1$ orbits.

Example 2

$G_3 = \langle g, h | (h^{-1}g)^3 \rangle.$