Finite quandle rings and matrices

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Brief history of quandle theory

Quandle theory is a relatively new subject in abstract algebra which has origins in knot theory and new applications to various other areas of mathematics currently being explored.
The history of this subject is a story of an idea which keeps getting reinvented and rediscovered. The earliest currently known example dates back to 1940s Japan when Mituhisa Takasaki defined *kei*, objects which were later known as involutory quandles [[1]].

Variants on the quandle idea have been studied by Conway (wracks), Brieskorn (automorphic sets), Matveev (distributive groupoids), and Kauffman (crystals), though the current terminology is due to David Joyce, who coined the word "quandle" in his 1980 doctoral dissertation.

[1] Takasaki, Mituhisa *Abstractions of symmetric functions*, Tohoku Math. J. **49** (1943), 143–207.

Quandles as invariants of knots

Quandles are generally non-associative algebraic structures. They were introduced independently in 1982 by Joyce [[2]] *(quandles)* and Matveev [[3]] *(distributive groupoids)* with the purpose of constructing invariants of knots.

[2] Joyce, David, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra. **23** (1982), 37-65.

[3] Matveev, S. V., *Distributive groupoids in knot theory*, Mat. Sb. **119** (1982), no. 1, 78–88. (Russian).

Definitions of quandles and racks

Definition

A quandle is a set Q with a binary operation $\triangleright \colon Q \times Q \to Q$ satisfying the three axioms

(i) for every $a \in Q$, we have $a \triangleright a = a$, *(idempotence)*

(ii) for every pair $a, b \in Q$ there is a unique $c \in Q$ such that $a = c \triangleright b$, *(right-invertibility)* and

(iii) for every $a, b, c \in Q$, we have $(a \triangleright b) \triangleright c = (a \triangleright c) \triangleright (b \triangleright c)$. *(self-distributivity)*

Definition

A rack is a set with a binary operation that satisfies axioms (ii) and (iii).

Examples of quandles

Quandle theory may be thought of as analogous to group theory. Indeed, groups are quandles with the quandle operation given by n-fold conjugation for an integer n.

Example (n-fold conjugation quandle)

Let G be a group and $n \in \mathbb{Z}$. Then the $(G, \triangleright)$ is a quandle, where

$$x \triangleright y = y^{-n} x y^{n} \text{ for all } x, y \in G.$$

Example ($Core(G)$ quandle)

Any group G with the quandle operation:

$$x \triangleright y = y x^{-1} y$$

is a quandle called $Core(G)$.

Examples of quandles

Example (Alexander quandle)

is a module M over the ring $R = \mathbb{Z}[t^{\pm 1}]$ of Laurent polynomials in one variable with quandle operation

$$x \triangleright y = tx + (1-t)y \text{ for all } x, y \in M.$$

Examples of quandles

Example (Dihedral quandle)

Let n be a positive integer, then for elements $k, m \in \mathbb{Z}_n$, define

$$k \triangleright m = 2m - k \pmod{n}.$$

Then "$\triangleright$" defines a quandle structure called the dihedral quandle, and denoted by R_n, that coincides with the set of reflections in the dihedral group with composition given by conjugation.

Integral quandle matrix

Let $Q = \{q_1, q_2, \dots, q_n\}$ be a finite quandle with n elements. We define the matrix of Q, denoted M_Q, to be the matrix whose entry in row i column j is $q_i \triangleright q_j$.

The matrix

$$M_Q = \begin{pmatrix} q_1 \triangleright q_1 & q_1 \triangleright q_2 & \dots & q_1 \triangleright q_n \\ q_2 \triangleright q_1 & q_2 \triangleright q_2 & \dots & q_2 \triangleright q_n \\ \dots & \dots & \ddots & \dots \\ q_n \triangleright q_1 & q_n \triangleright q_2 & \dots & q_n \triangleright q_n \end{pmatrix}$$

is really just the quandle operation table considered as a matrix, with the columns acting on the rows. In particular, if the elements of the quandle are the numbers $Q = \{1, 2, \dots, n\}$ with $M_Q = [\alpha_{ij}]$, call M_Q an *integral quandle matrix*.

Example of quandle matrix

Let $T_n = \{1, 2, \ldots, n\}$ be the trivial quandle of order n then its matrix is

$$M_{T_n} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 2 & 2 & \ldots & 2 \\ . & . & \ddots & . \\ n & n & \ldots & n \end{pmatrix}.$$

Standard form of integral quandle matrix

If the entries on the diagonal in an integral quandle matrix are in the usual order, i.e., $\alpha_{ii} = i$, then $i \triangleright j$ is just the entry in row i column j.

An integral quandle matrix of this type is in *standard form*.

The quandle axioms place certain restrictions on what kind of matrices can arise from a quandle.

Lemma (Ho, Benita and Nelson, Sam)

The matrix $M_Q = [\alpha_{ij}]$ is a integral matrix of the finite quandle Q (in standard form) if and only if the following conditions are satisfied:

Standard form of integral quandle matrix

(i) *(Idempotence)* The diagonal entries are distinct and $\alpha_{ii} = i$, where $i \in \{1, 2, \ldots, n\}$.

(ii) *(Right-invertibility)* The entries in each column are distinct, i.e. $\alpha_{ij} = \alpha_{kj}$ implies $i = k$.

(iii) *(Self-distributivity)* If we denote $\alpha_{ij} = a[i,j]$, then the entries must satisfy equality $a[a[i,j],k] = a[a[i,k],a[j,k]]$.

Isomorphism classes of finite quandles

In [4] the integral matrices were used for distinguishing all isomorphism classes of finite quandles with up to 5 elements and computing the automorphism group for each quandle.

Theorem (4, Theorem 4)

Two integral quandle matrices in standard form determine isomorphic quandles iff they are p-equivalent by a permutation $\rho \in S_n$.

The permutation $\rho \in S_n$ determine the p-equivalent (permutation-equivalent) matrix $\rho(M_Q)$ to the integral quandle matrix M_Q by the rule $\rho(M_Q) = A_\rho^{-1}(\rho(\alpha_{ij}))A_\rho$, where A_ρ is the permutation matrix of ρ.

[4] Ho, Benita and Nelson, Sam, *Matrices and Finite Quandles*, Homology, Homotopy and Applications. **7** (2005), no. 1, 197–208.

Involutory and self-dual quandles

The uniqueness in axiom (ii) of quandles implies that the map $f_b : Q \to Q$ defined by $f_b(a) = a \triangleright b$ is a bijection; the inverse map f_b^{-1} then defines the dual operation $a \triangleleft b = f_b^{-1}(a)$.

The set Q then forms a quandle $Q^* = (Q, \triangleleft)$ under $\triangleleft$, called the dual of $(Q, \triangleright)$.

Definition

If the dual quandle operation is the same as original quandle operation, i.e., $a \triangleleft b = a \triangleright b$ or $(a \triangleright b) \triangleright b = a$ for all $a, b \in Q$, then such quandles are called *involutory* (since all the maps $f_b : Q \to Q$ are involutions).
We will call a quandle Q *self-dual* if $Q \cong Q^*$.

Example of non self-dual quandle

Example

Let $Q = \{1,2,3,4,5\}$ be a quandle with integral quandle matrix

$$M_Q = \begin{pmatrix} 1 & 3 & 4 & 5 & 2 \\ 3 & 2 & 5 & 1 & 4 \\ 4 & 5 & 3 & 2 & 1 \\ 5 & 1 & 2 & 4 & 3 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}.$$

Then the dual quandle Q^* has matrix

$$M_{Q^*} = \begin{pmatrix} 1 & 4 & 5 & 2 & 3 \\ 5 & 2 & 4 & 3 & 1 \\ 2 & 1 & 3 & 5 & 4 \\ 3 & 5 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 & 5 \end{pmatrix}.$$

Since $M_Q \neq M_{Q^*}$ then quandle Q is not involutory and there are no permutations $\rho \in S_5$ such that $\rho(M_{Q^*}) = M_Q$, i.e. $Q \ncong Q^*$, so the quandle Q is not a self-dual.

Classification of self-dual quandles

It is easy to check that there is only one quandle of order 1 and one quandle of order 2, both trivial (i.e., $x \triangleleft y = x, \forall x, y \in Q$.)

Proposition

(i) *There are three quandle isomorphism classes of order 3 and they all are involutory (self-dual) quandles, one of them is commutative.*

(ii) *There are seven isomorphism classes of quandles of order 4, the five of them are involutory and two others are self-dual but not involutory.*

(iii) *There are 22 isomorphism classes of quandles of order 5. There are pair of non self-dual quandles and one commutative quandle of order 5.*

Computational results

Using techniques from [5], [6] and applying packages GAP[7], RiG[8] we compute matrices of finite involutory, self-dual quandles and make the corresponding classification of quandles of low order.

[5]Ho, Benita and Nelson, Sam, *Matrices and Finite Quandles*, Homology, Homotopy and Applications. **7** (2005), no. 1, 197–208.

[6]Lopes, Pedro and Roseman, Dennis, *On Finite Racks and Quandles*, Communications in Algebra. **34** (2006), no. 1, 371–406.

[7]The GAP Group, *GAP – Groups, Algorithms, and Programming*, Version 4.10.2; 2019. (https://www.gap-system.org)

[8]Vendramin, Leandro *Rig, a GAP package for racks, quandles and Nichols algebras.* Available at http://code.google.com/p/rig/

Definition of Quandle Rings

Let $(Q, \triangleright)$ be a quandle and $(R, +, \cdot)$ be an associative ring with identity.

Following [9], [10] we can consider a ring $R[Q]$ (non-associative in general) as the set of all formal finite R-linear combinations of elements of quandle Q, that is,

$$R[Q] = \left\{ \sum_{q \in Q} r_q q \mid r_q \in R \text{ and } r_q = 0 \text{ for almost all } q \in Q \right\}$$

[9] Bardakov, Valeriy G., Passi, Inder Bir S. and Singh, Mahender *Quandle rings*, Journal of Algebra and Its Applications. **18** (2019), no. 8, 1950157 (23 pages).

[10] Elhamdadi, Mohamed, Fernando, Neranga and Tsvelikhovskiy, Boris *Ring theoretic aspects of quandles*, Journal of Algebra. **526** (2019), 166–187.

Definition of Quandle Rings

with the natural addition

$$\left(\sum_{x\in Q} a_x x\right) + \left(\sum_{y\in Q} b_y y\right) = \sum_{x\in Q}(a_x + b_x)x,$$

and the multiplication given by the following

$$\left(\sum_{x\in Q} a_x x\right)\left(\sum_{y\in Q} b_y y\right) = \sum_{x,y\in Q} a_x b_y (x \triangleright y),$$

where $x, y \in Q$, $a_x, b_y \in R$.

Augmentation ideal

Analogous to group rings, we can define the augmentation map $\alpha : R[Q] \to R$ by setting $\alpha\left(\sum_{q\in Q} r_q q\right) = \sum_{q\in Q} r_q$.
Clearly, α is a surjective ring homomorphism, and

$$\Delta_{R(Q)} := \ker(\alpha)$$

is a two-sided ideal of $R[Q]$, called the *augmentation ideal* of $R[Q]$. Thus, we have $R[Q]/\Delta_{R(Q)} \cong R$ as rings.

Associated matrices

Analogous to group rings, we can rewrite the product of two elements $a = \sum_{i=1}^{n} \alpha_i q_i$ and $b = \sum_{i=1}^{n} \beta_i q_i$ of a quandle ring $R[Q]$, where $Q = \{q_1, q_2, \ldots, q_n\}$ is a finite quandle, in the following way

$$ab = \sum_{1 \le i,j \le n} \alpha_i \beta_j (q_i \triangleright q_j) = \sum_{i=1}^{n} \left(\sum_{1 \le j \le n,\ q_i = q_k \triangleright q_j} \alpha_k \beta_j \right) q_i.$$

Since for any $i, j \in \{1, 2, \ldots, n\}$ the quandle equation $q_i = q_k \triangleright q_j$ has unique solution, that is $q_k = q_i \triangleleft q_j$, we can associate the matrix $A = [a_{i,j}]_{1 \le i,j \le n}$ over ring R to the element $a \in R[G]$, where $a_{i,j} = \alpha_k$. Thus, the computation of product of elements $a, b \in R[Q]$ is a simple matrix multiplication of the matrix A by the column $(\beta_1, \beta_2, \ldots, \beta_n)$.

Examples of Associated matrices

Let $Q = \{q_1, q_2, q_3, q_4\}$ be a quandle with integral quandle matrix

$$M_Q = \begin{pmatrix} 1 & 4 & 2 & 3 \\ 3 & 2 & 4 & 1 \\ 4 & 1 & 3 & 2 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$ and ring $R = \mathbb{Z}_6$.

Then the dual quandle Q^* has matrix $M_{Q^*} = \begin{pmatrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \end{pmatrix}$

which is transpose to matrix M_Q incidently.
Since $M_Q \neq M_{Q^*}$ then quandle Q is not involutory, but exists permutation $\rho = (34)$ such that $\rho(M_{Q^*}) = M_Q$, i.e. $Q \cong Q^*$, so the quandle Q is a self-dual.

Examples of Associated matrices

Let $a = q_1 + 5q_2 + 4q_3 + 2q_4$ be some element of quandle ring $\mathbb{Z}_6[Q]$. The element a belongs to the augmentation ideal $\Delta_{\mathbb{Z}_6[Q]}$ also.

Since the dual quandle Q^* has matrix $M_{Q^*} = \begin{pmatrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \end{pmatrix}$

then $A = \begin{pmatrix} 1 & 4 & 2 & 5 \\ 2 & 5 & 1 & 4 \\ 5 & 2 & 4 & 1 \\ 4 & 1 & 5 & 2 \end{pmatrix}$ is associated matrix of the element $a \in \mathbb{Z}_6[Q]$.

Some ongoing problems

- To discover correlations between properties of elements $a \in R[Q]$ and their associated matrices $A \in M_n(R)$.
- To describe all left divisors $a \in R[Q]$ of fixed element of quandle ring (algebra) in term of their associated matrices $A \in M_n(R)$.
- If quandle Q is algebraically connected or just connected, i.e. it has only one orbit under the inner automorphism group, then which properties must satisfy ring R for connectivity of a quandle ring $R[Q]$?

The last slide

Thank you for attention!