Some Groups Arising from Wild Topology

W. Herfort

(Wien, University of Technology)

Groups and Their Actions

Gliwice, Silesian University of Technology

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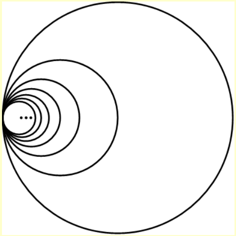
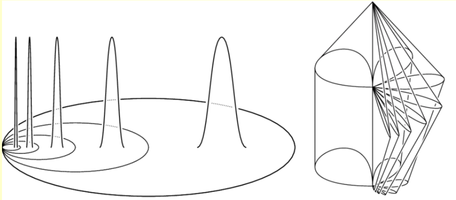
Two "Animals", both in their own way wild



Hawaiian Earring (HE) (See Definition 13)



Harmonic Archipelago (HA)

Topologist's Product

G. Higman (1952), H. B. Griffiths (1954), B. de Smit (1992), K. Eda (1992) and J. Cannon & G. Conner (2000) agree on the following definition of the *topologist's product* (=topological free product) $\circledast_i G_i$ of a given a sequence $(G_i)_{i\geq 1}$ of groups.

- Form, for $n \geq 1$, the free product $F_n := *_{i=1}^n G_i$.
- Consider the inverse system $p_n : F_{n+1} \to F_n$ where p_n has kernel the normal closure of G_{n+1}.
- Let $\hat{G} = \varprojlim_n F_n$.
- A *legal word* in $\hat{G}$ is a sequence $(f_n)_{n\geq 1}$ such that for any given $j \geq 1$ the number of times a nontrivial element $g_j \in G_j$ appears in the reduced word presentation of f_n is bounded independently of n.
- The set of legal words forms a subgroup of $\hat{G}$, the *topologist's product*, denoted by $\circledast_i G_n$.

Some Properties of $P := \circledast_i \mathbb{Z}$

When $G_i \cong \mathbb{Z}$ we shall write P for $\circledast_{i \geq 1} \mathbb{Z}$.

- The inverse limit $\hat{G}$ corresponds to the Čech fundamental group of the HE.
- P is *not* free and every finitely generated subgroup is free (P is *locally free*). (G. Higman 1952)
- P is the fundamental group of a shrinking wedge of circles. (Griffiths 1954, correction of proof by Morgan & Morrison 1986, de Smit 1992)

Cotorsion of Abelian Groups (e.g. book of L. Fuchs 2015)

Let A be an abelian group.

- The group A is *cotorsion* if, and only if, whenever $A \leq G$ and G/A is torsion-free then A is a direct summand.
- Examples: $\mathbb{Q}$, p-adic numbers $\mathbb{Z}_p$, finite groups, $(\mathbb{R} \oplus \hat{\mathbb{Z}})^{\mathbb{N}}$.

Higman Completeness

Definition

The group G is *Higman-complete* provided for every sequence $(f_i)_{i \geq 1}$ of nontrivial elements in G there is a solution sequence for the infinite system of equations

$$h_{i-1} = w_i(f_i, h_i), \quad i \geq 1.$$

- G abelian is Higman-complete if, and only if, G is cotorsion.
- Higman-completeness is inherited by factor groups.

(H. & Hojka, 2017)

Abelian Groups: Slenderness

Definition

The group A is *slender* provided every homomorphism $h : \mathbb{Z}^{\mathbb{N}} \to A$ factors through a finite projection.

Example: $A = \mathbb{Z}$ (Nunke 1961).

n-Slenderness

Definition

The group G is *n-slender* provided every homomorphism $h : \circledast_{i\geq 1} \mathbb{Z} \to G$ factors through a projection $p_n : \circledast_{i\geq 1} \mathbb{Z} \to F_n$.

- $\mathbb{Z}$ is n-slender. (Higman 1952)
- Every slender group is n-slender.
  (Eda 1992, de Smit 1992)
- Every word hyperbolic torsion-free group is n-slender. Graph products of n-slender groups are n-slender.
  (S. Corson 2015)

T-Kernel of an n-Slender Group T

For a group G and an n-slender group T let the *T-kernel* be defined as

$$\mathrm{Ker}_T(G) := \bigcap \{\ker(\phi) : \phi \in \hom(G, T)\}.$$

- $\mathrm{Ker}_{\mathbb{Z}}(F_n) = F_n'$, the commutator subgroup.
- $\mathrm{Ker}_{\mathbb{Z}}(\mathbb{Q}) = \mathbb{Q}$, because $\mathbb{Q}$ is divisible and cannot map onto $\mathbb{Z}$.
- One has $\circledast_{i\geq 1}\mathbb{Z}/\,\mathrm{Ker}_{\mathbb{Z}}(\circledast_{i\geq 1}\mathbb{Z}) \cong \mathbb{Z}^{\mathbb{N}}$.
  (Eda 1992, Cannon & Conner 2000)
- $\mathrm{Ker}_{B(1,n)}(F_n) = F_n''$, the second derived group. Here $B(1,n)$ is a Baumslag-Solitar group and $n \neq 0 \pm 1$. (A basis theorem for the commutator subgroup of a free metabelian group, due to W. Tomaszewski, is used, 2003).
  (Conner & Kent & H. & Pavešić 2018)
- $(\circledast_{i\geq 1}\mathbb{Z})\,\mathrm{Ker}_{\mathbb{Z}}(\hat{G}) = \hat{G}$. This equation can be used to construct a path-connected fibration with base the HE, fundamental group $\mathrm{Ker}_{\mathbb{Z}}(\circledast_{i\geq 1}\mathbb{Z})$, and profinite fibers.
  (Conner & Kent & H. & Pavešić 2019)

Abelian Groups: Chase's Lemma

Theorem

(Chase's Lemma, 1962) Fir every every homomorphism

$$h : \prod_{i \geq 1} A_i \to \bigoplus_{i \geq 1} B_i$$

there exist $k, m, n \geq 1$ *such that*

$$h(m \prod_{i \geq k} A_i) \leq \bigoplus_{i \leq n} B_i + U(\bigoplus_{i \geq 1} B_i),$$

where $U()$ *is the Ulm subgroup.*

Chase's Lemma: Splitting as a Free Product

- Eda 2011: Any homomorphism

$$\phi : \circledast_{i\geq 1} G_i \to A * B :$$

either factors through a canonical projection onto $*_{1\leq i\leq n} G_i$ or, for some $k \geq 1$, the image of $\circledast_{j\geq k} G_j$ under ϕ is, up to conjugation, contained in a free factor.
Consequence: The only free factorization of $G = \circledast_{i\geq 1} G_i$ with G_i freely indecomposable is to split off a $*_{i=1}^{n} G_i$ for some $n \geq 1$.

- K. Eda 1998: Any homomorphism

$$\phi : \varprojlim_{i\geq 1} G_i \to \circledast_{i\geq 1}\mathbb{Z}$$

either factors through a canonical projection $p_n : \hat{G} \to G_n$ or the image under ϕ belongs, up to conjugation, to one of the free factors $\mathbb{Z}$.

Hulanicki, Kaplansky, Balcerzyk: Structure Theorems for $\prod_{i\geq 1} G_i / \bigoplus_{i\geq 1} G_i$

Theorem

(I. Kaplansky 1957, A. Hulanicki 1958) Let $(G_n)_{n\geq 1}$ be a sequence of torsion-free abelian groups. Then the factor group $\prod_{i\geq 1} G_i / \bigoplus_{i\geq 1} G_i$ is algebraically compact.

Theorem

(A. Hulanicki, 1958) A reduced abelian group is algebraically compact if, and only if, it is the complete direct sum of finite cyclic groups and p-adic groups $\mathbb{Z}_p$ for p running through a set of primes.

Theorem

(S. Balcerzyk, 1959) The group $\prod_{i\geq 1} \mathbb{Z} / \bigoplus_{i\geq 1} \mathbb{Z}$ is isomorphic to $\mathbb{Z}^{\mathbb{N}} \oplus \prod_p A_p$ and $A_p \cong \mathbb{Z}_p^{\mathbb{N}}$.

An Analogue: Archipelago Groups

The *Archipelago group* is defined as $\mathcal{A}(G_i) := \circledast_i G_i/N$ for N the normal closure of $\bigcup_i G_i$ in $\circledast_i G_i$. See (2).

- $\mathcal{A}(G_i)$ is Higman-complete, locally free, and freely indecomposable. (H. & Hojka, 2017, 2019)
- $\mathcal{A}(G_i)$ contains a copy of every countable locally free group. (W. Hojka, 2017)
- The abelianization of $\mathcal{A}(G_i)$ is cotorsion. (H. & Hojka, 2017)
- The abelianization of $\mathcal{A}(G_i)$ is isomorphic to $\prod_{i\geq 1} \mathbb{Z}/\bigoplus_{i\geq 1} \mathbb{Z} \cong (\mathbb{R} \oplus \hat{\mathbb{Z}})^{\mathbb{N}}$. (K. Eda 2000)

Free Subgroups of the Baer-Specker Group

Theorem

Let B be the subgroup of bounded functions in $\mathbb{Z}^{\mathbb{N}}$. Then B is a free abelian subgroup of $\mathbb{Z}^{\mathbb{N}}$.

(E. Specker 1950, G. Nöbelung 1968)

Free Subgroups of HEG

Theorem

The subgroup of all sequences (f_n) such all g_j occur at most a fixed number of times, is free.

(A. Zastrow, 1997, generalizations 2003, Eda 1999).

Theorem

Any Archipelago group $G := \mathcal{A}(G_i)$ contains a subgroup T with

(a) $T \cong \ast_{\mathfrak{c}} \mathbb{Q}$.

(b) $TG'/G' \cong \bigoplus_{\mathfrak{c}} \mathbb{Q}$.

(H. & Hojka 2017)

Abelianization of $G := \circledast_{i\geq 1} G_i$

We let $G = \circledast_{i\geq 1} G_i$.

- If $G_i \cong \mathbb{Z}$ then $G/G' \cong \mathbb{Z}^{\mathbb{N}} \oplus (\mathbb{R} \oplus \hat{\mathbb{Z}})^{\mathbb{N}}$. (Eda & Kawamura, 2000)
- If $G_i \cong \mathbb{Z}(p)$ for p a prime then $G/G' \cong \left(\bigoplus_{i\geq 1} \mathbb{Z}(p)\right) \bigoplus (\mathbb{R} \oplus \hat{\mathbb{Z}})^{\mathbb{N}}$. (H. & Hojka, 2017)

Outlook / Announcements

- (Conner & H. & Kent & Pavešić, 2020) Letting $F_c(x,y)$ be the free class c nilpotent group on two generators then

$$(\circledast_{i\geq 1}\mathbb{Z})\,\mathrm{Ker}_{F_2(x,y)}(\hat{G}) = \hat{G}, \quad (\circledast_{i\geq 1}\mathbb{Z})\,\mathrm{Ker}_{F_3(x,y)}(\hat{G}) < \hat{G}.$$

- (H. & Hojka, 202?) When $G = \circledast_{i\geq 1} G_i$ and all G_i are torsion-free abelian groups then

$$G/G' \cong (\prod_{i\geq 1} G_i) \oplus (\prod_{i\geq 1}\mathbb{Z}/\bigoplus_{i\geq 1}\mathbb{Z}).$$

- (H. & Hojka, 202?) Let G be a group and $(H_i)_{i\geq 1}$ be a decreasing sequence of subgroups. G is generalized Higman complete if given $f_i \in H_i$ and $w_i \in F(x,y)$ then the system $h_{i-1} = w_i(f_i, h_i)$, $i \geq 1$, has a solution sequence with $h_i \in H_i$. When G is generalized Higman complete then for every homomorphism $h : G \to A * B$ there is $k \geq 1$ with $h(H_k) \leq A$ or $h(H_k) \leq B$ up to conjugation in $A * B$. This result implies the splitting results of K. Eda from 2011.

Dziękuję bardzo!

THANK YOU FOR YOUR ATTENTION.