Some words on simplicity and amenability of metric ultraproducts

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Work in progress ...

Bi-invariant norm and invariant metric on a group

Suppose G is a group.

$d\colon G \times G \to \mathbb{R}_{\geqslant 0}$ is an *invariant metric* on G, when

$$d(gx, gy) = d(x, y) = d(xg, yg)$$

for all $g, x, y \in G$

Each such invariant metric comes from a *bi-invariant (i.e. conjugacy invariant) norm (lenght)* $\|\cdot\|\colon G \to \mathbb{R}_{\geqslant 0}$ (another notation $\ell\colon G \to \mathbb{R}_{\geqslant 0}$) satisfying

- $\|gh\| \leqslant \|g\| + \|h\|$
- $\left\|g^{-1}\right\| = \|g\| = \|hgh^{-1}\|$
- $\|g\| = 0$ if and only if $g = e$ (pseudonorm, when only $\|e\| = 0$)

$$\|\cdot\| \quad \rightsquigarrow \quad d(x, y) = \left\|xy^{-1}\right\|$$

$$d(\cdot,\cdot) \quad \rightsquigarrow \quad \|g\| = d(g, e)$$

Examples of norms and lengths

Examples of bounded and unbounded norms

- Discrete norm: $\|g\| := \begin{cases} 1 & : g \neq e \\ 0 & : g = e \end{cases}$
- Hamming norm S_n: $\sigma \in S_n$, $\|\sigma\|_H := \|\{i : \sigma(i) \neq i\}\|$
- Rank norm on $\mathrm{GL}_n(F)$ (F: field) $\|g\|_r := rank(g - I)\,(= \dim(Im(g - I)))$
- Conjugacy length (pseudonorm) on a finite group G:

$$\ell_c(g) := \frac{\log |g^G|}{\log |G|}$$

  it is a norm when $Z(G) = \{e\}$
- Invariant word norm of a group G: let $S = S^{-1} \subseteq G$ be a normal subset (that is $s \in S \to s^x = x^{-1}sx \in S$)

$$\|g\|_S = \min\{n : g \text{ is a product of } n \text{ conjugates of elements from } S\}.$$

Metric ultraproduct

Let $\mathcal{G} = (G_m, \|\cdot\|_m)_{m\in\mathbb{N}}$ be a family of metric groups and choose a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$

Definition

Metric ultraproduct

$$G^*_{\mathrm{met}} = \prod_{m\in\mathbb{N}}^{\mathrm{met}} G_m = G_{\mathrm{fin}}/N_{\mathcal{U}}$$

where

$$G_{\mathrm{fin}} = \left\{ (g_m) \in \prod_{m\in\mathbb{N}} G_m : \sup_{m\in\mathbb{N}} \|g_m\|_m < \infty \right\} \text{ oraz } N_{\mathcal{U}} = \left\{ (g_m) : \lim_{m\to\mathcal{U}} \|g_m\|_m = 0 \right\}$$

G^*_{met} is a topological group. Topology comes from a canonical invariant norm

$$\|\cdot\|\colon G^*_{\mathrm{met}} \to \mathbb{R}_{\geqslant 0} \text{ defined by } \|(g_m)/N_{\mathcal{U}}\| = \lim_{m\to\mathcal{U}} \|g_m\|_m.$$

Examples of simple metric ultraproduct

$\mathcal{S} = \prod_{m\in\mathbb{N}}^{\text{met}}(S_m, \frac{1}{n}\|\cdot\|_H)$ – ultraproduct of S_n with normalised Hamming norms ($\mathcal{S}$ is a universal sofic group, big open problem: is every fin. gen. group, a subgroup of $\mathcal{S}$?)

Theorem (G. Elek - E. Szabó, Math. Ann. (2005))

$\mathcal{S}$ is a simple group

Let $\{(G_m, \ell_c)\}_{m\in\mathbb{N}}$ be a family of finite simple groups, where $\ell_c(g) = \frac{\log|g^G|}{\log|G|}$.

Theorem (Nikolov arXiv 2009, Stolz - Thom, PLMS 2014, Ivanov, arXiv 2014)

1. $\prod_{m\in\mathbb{N}}^{met}(G_m, \ell_c)$ *is a simple group*
2. *Metric ultraproduct of centerless projective classical groups (e.g. PGL) over finite fields is a simple group*

The proof is based on a deep result by M. W. Liebeck and A. Shalev: for $g \in G$, $N \in \mathbb{N}$ let $C_N(g) = \left(g^G \cup {g^{-1}}^G\right)^{\leqslant N}$ and $C_0(g) = \{e\}$

Theorem (M. W. Liebeck & A. Shalev, Annals of math. 2001)

There is a constant $L > 0$, such that for every non-abelian finite simple group G and $g \in G$, $G = C_{L/\ell_c(g)}(g) = \left(g^G \cup {g^{-1}}^G\right)^{\leqslant L/\ell_c(g)}$

When standard ultraproduct is simple?

In case of discrete norm, we have the following easy criterion

Fact

$\prod_{m\in\mathbb{N}} G_m/\mathcal{U}$ is simple $\Leftrightarrow$ there is $N \in \mathbb{N}$, such that for $\mathcal{U}$-almost $m \in \mathbb{N}$ group G_m is N-uniformly simple, that is $G_m = C_N(g)$ for all $g \in G_m, g \neq e$.

$$C_N(g) = \left(g^G \cup {g^{-1}}^G\right)^{\leqslant N}$$

Uniformly simple groups:

Theorem

1. *(JG, A. Muranov, 2013) Automorphism group of a bi-regular tree is 8-uniformly simple*
2. *(ŚRGal, JG, 2017) simple Higman-Thompson groups, commutator subgroup $[F, F]$ of Thompson F are 6-uniformly simple*
3. *(P. Dowerk, A. Thom 2018, 2019) Finite-dimensional projective unitary groups, as well as the projective unitary groups of Type II_1 factors are uniformly simple*

When metric ultraproduct is simple? Bounded norm case

Let us define, for a bounded metric group $(G, \|\cdot\|)$, *big sequences* of positive reals.

Definition

For $r > 0$, sequence $(\varepsilon_0, \ldots, \varepsilon_N)$ from $\mathbb{R}_{>0}$ is *r-big*, if for all $g \in G$ with $\|g\| > r$ we have

$$G = C_0(g)B_{\varepsilon_0}(e) \cup \ldots \cup C_N(g)B_{\varepsilon_N}(e),$$

where $B_\varepsilon(g) = \{h \in G : \|gh^{-1}\| \leqslant \varepsilon\}$ is an ε-ball around g, $C_N(g) = \left(g^G \cup {g^{-1}}^G\right)^{\leqslant N}$

When metric ultrapower is simple?

Theorem (JG, K. Majcher, M. Ziegler)

Let $(G, \|\cdot\|)$ be a metric group with bounded metric (i.e. $\|\cdot\| < c$ for some constant $c > 0$). Then metric ultrapower G^*_{met} of G is simple $\Leftrightarrow$ for all $r > 0$ every infinite sequence $(\varepsilon_0, \ldots, \varepsilon_n, \ldots)$ from $\mathbb{R}_{>0}$ has some r-big finite initial segment.

Corollary

$\prod (IET, \mu)^{met}$ *is simple*

Simple metric ultraproduct

$(\varepsilon_0, \dots, \varepsilon_N) \subset \mathbb{R}_{>0}$ is *r-big* $\Leftrightarrow$ $g \in G, \|g\| > r, \; G = C_0(g)B_{\varepsilon_0}(e) \cup \ldots \cup C_N(g)B_{\varepsilon_N}(e)$

Theorem (JG, K. Majcher, M. Ziegler)

Let $(G_m, \|\cdot\|_m)_{m\in\mathbb{N}}$ be a family of uniformly bounded groups (i.e. $\|\cdot\|_m < c$ for some $c > 0$). Then $\prod_{m\in\mathbb{N}}^{met} G_m$ is simple $\Leftrightarrow$ for every $r > 0$ and every infinite sequence $(\varepsilon_0, \dots, \varepsilon_n, \dots) \subset \mathbb{R}_{>0}$ there is $N \in \mathbb{N}$, such that for $\mathcal{U}$-almost all $m \in \mathbb{N}$, sequence $(\varepsilon_0, \dots, \varepsilon_N)$ is r-big for G_m.

For permutations $\{S_n, \frac{1}{n}\|\cdot\|_H\}_{n\in\mathbb{N}}$ this number $N := \max\left\{\frac{8}{r}, \frac{2}{\varepsilon_{\frac{8}{r}}}\right\}$, due to the following result of Brenner 1978: if $\sigma \in A_n$ is nonexceptional and fixed-point-free (i.e. $\|\sigma\|_H = n$), then $A_n = C_4(\sigma)$

For $r > 0$ let

$$T_r = \{r - \text{small finite sequences}\}.$$

T_r is closed under initial subsequences, so is a *set-theoretic tree*. Then, the condition from the theorem reads as

> *G^*_{met} is simple $\Leftrightarrow$ for all $r > 0$, T_r has no infinite long path, that is T_r is a well founded tree.*

What are possible ranks of such trees, for simple metric ultraproducts?

We say that a metric group $(G, \|\cdot\|)$ is *metrically simple*, iff G^*_{met} is simple

Metric simplicity implies topological simplicity (lack of proper normal closed subgroups).

Fact

Every simple compact metric group is metrically simple.

Because usually $G^*_{\text{met}} \cong G$.

What are trees $\{T_r\}_{r>0}$ for simple compact metric groups?

When metric ultraproduct of unbounded group is simple?

For an unbounded metric group $(G, \|\cdot\|)$, we have the following result

When metric ultrapower is simple?

Theorem (JG, K. Majcher, M. Ziegler)

Let $(G, \|\cdot\|)$ be a metric group, possibly with unbounded metric. Then metric ultrapower G^*_{met} of G is simple $\Leftrightarrow$ for all $r > 0$ and $t > 0$, for every infinite sequence $(\varepsilon_0, \ldots, \varepsilon_n, \ldots) \subset \mathbb{R}_{>0}$, there is $N \in \mathbb{N}$ such that for all $g \in G$, $r < \|g\| < t$

$$B_t(e) \subseteq C_0(g)B_{\varepsilon_0}(e) \cup \ldots \cup C_N(g)B_{\varepsilon_N}(e).$$

Corollary

The following groups $\prod_{m\in\mathbb{N}}^{met}(A_m, \|\cdot\|_H)$ and $\prod_{m\in\mathbb{N}}^{met}(A_\infty, \|\cdot\|_H)$ are simple.

Metric ultrapower of linear groups (Chevalle groups) with ranks norm over algebraically closed fields

K_m - field, $G_m = \mathrm{PSL}_m(K_m)$ (more generally $G_m(K_m)$ – simple centerless Chevalley group $(Z(G_m(K_m)) = \{e\})$)

When each K_m is simple,

$$\prod_{m\in\mathbb{N}}^{\text{met}} (G_m, \ell_c)$$

is simple, where $\ell_c(g) := \frac{\log |g^G|}{\log |G|}$ (Stolz - Thom, '14, Nikolov '09).

How about infinite fields? One need to consider another norm, e.g. rank norm

$$\|g\|_r := \frac{1}{m} rank(g - I) = \frac{1}{m} \dim(Im(g - I)).$$

We conjecture, that for all fields K_m, metric ultrapower $\prod_{m\in\mathbb{N}}^{\text{met}} (G_m(K_m), \|\cdot\|_r)$ is simple.

Theorem (JG, KM, MZ)

If each field K_m is algebraically closed, then $\prod_{m\in\mathbb{N}}^{\text{met}} (G_m(K_m), \|\cdot\|_r)$ is simple.

Idea of the proof: ℓ_c and $\|\cdot\|_r$ are asymptotically equivalent, use Liebeck-Shalev result for the rank norm, which is first order expressible

Amenability

G is uniformly amenable[1] $\Leftrightarrow$ standard ultrapower $G^{\mathbb{N}}/\mathcal{U}$ is amenable $\Leftrightarrow$ G satisfies the uniform Følner condition: for every $\varepsilon > 0$ and $N \in \mathbb{N}$ there is $M \in \mathbb{N}$ such that for every finite $X \subset G$, $|X| < N$ there is finite $\emptyset \neq Y \subset G$, $|Y| < M$ such that $|XY| < (1+\varepsilon)|Y|$
Every abelian[2], nilpotent and solvable group is uniformly simple

When metric ultraproduct $\prod_{m\in\mathbb{N}}^{\text{met}}(G_m, \|\cdot\|_m)$ is an amenable group as a discrete group? Uniform metric Følner condition?

Using recent ideas on topological matchings and amenability of A. Thom and F. M. Schneider[3] we have

Fact (JG, KM, MZ)

suppose $(G_m, \|\cdot\|_m)_{m\in\mathbb{N}}$ *is a family of metric groups with uniformly bounded groups. Then* $\prod_{m\in\mathbb{N}}^{met}(G_m, \|\cdot\|_m)$ *is amenable as a discrete group* $\Leftrightarrow$ *for all* $r > 0, \Theta > 0, n \in \mathbb{N}$ *and every infinite* $(\varepsilon_0, \dots, \varepsilon_n, \dots) \subset \mathbb{R}_{>0}$ *there is* $N \in \mathbb{N}$*, such that* $(\varepsilon_0, \dots, \varepsilon_N)$ *is* (r, Θ, n)*-good for* $\mathcal{U}$*-almost all* $m \in \mathbb{N}$*.*

Where goodness refers to some kind of topological matchings ....

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[1] Marek Bożejko *Uniformly amenable discrete groups* Math. Ann. 251 (1980)

[2] J. Dronka, B. Wajnryb, P. Witowicz, K. Orzechowski *Growth functions for some uniformly amenable groups* Open Math. 15 (2017)

[3] F.M. Schneider, A. Thom *On Følner sets in topological groups* Compos. Math. 154 (2018)

Alternative for metric ultraproducts

Recall: G is N-uniformly simple, if $G = C_N(g)$ for all $g \in G, g \neq e$.

Theorem (ŚRGal, JG Trans. Amer. Math. B 4 (2017))

$[F, F]$ is 6-uniformly simple, where F is a Thompson group, that is 2-adic, piecewise linear, order preserving homeomerphisms of $[0, 1]$. More generally: suppose G acts on an infinite linear order $(I, \leqslant)$ by authomorphisms of bounded supports. If the action is proximal (i.e. for every $a < b$ and $c < d$ from I, there is $g \in G$, such that $g(a) < c < d < g(b)$), then $[G, G]$ is 6-uniformly simple.

Theorem (JG, KM, MZ)

Suppose G has no laws, and $[G, G]$ is uniform simple. Consider

$$\{(G, \|\cdot\|_m)\}_{m\in\mathbb{N}}$$

with possibly different uniformly bounded norms $\|\cdot\|_m$. Then metric ultraproduct $\prod_{m\in\mathbb{N}}^{met}(G_m, \|\cdot\|_m)$ is

abelian or contains F_2.

Corollary (JG, Klaudia Weigel)

Amenable metric ultrapowers of F are abelian.