Amenability and Computability II

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Groups and their actions 2019

Let G be a group, $D \subset\subset G$ and $n \in \mathbb{N}$.
A subset $F \subset\subset G$ is an n-**Følner set** with respect to D if

$$\forall x \in D \quad \frac{|F \setminus xF|}{|F|} \leq \frac{1}{n}.$$

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Definition. The binary function

$$F\o l_G(n, D) = min\{|F| : F \subseteq G \text{ such that } F \in \mathfrak{F}\o l_{G,D}(n)\}$$

where the variable D corresponds to finite sets, is called the **Følner function of** G.

Let $\nu : \mathbb{N} \to G$ be a numbering of a group G such that G is **computably enumerable** (i.e. the graphs of the equality, the multiplication and the inversion are computably enumerable).

Theorem

Let G be a computably enumerable group. The following conditions are equivalent:

1. *G is amenable;*
2. *the Følner function of G is* ***subrecursive****, i.e. admits a computable total upper bound;*
3. *G is Σ-amenable.*

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Moreover, computable amenability of G implies computability of it.

(i.e. the graphs of the equality, the multiplication and the inversion are decidable;
ν can be taken to be 1-1)

Computable amenability

Definition. The group (G, ν) is Σ-**amenable** if there exists an algorithm which for all pairs (n, D), where $n \in \mathbb{N}$ and $D \subset\subset \mathbb{N}$, finds a set $F \subset\subset \mathbb{N}$ containing a subset F', such that $\nu(F') \in \mathfrak{F}\o l_{G,\nu(D)}(n)$.

Definition. (Cavaleri[1]) The group (G, ν) is **computably amenable** if there exists an algorithm which for all pairs (n, D), where $n \in \mathbb{N}$ and $D \subset\subset \mathbb{N}$, finds a set $F \subset\subset \mathbb{N}$ such that $\nu(F) \in \mathfrak{F}\o l_{G,\nu(D)}(n)$ and $|F| = |\nu(F)|$.

[1] M. Cavaleri, Følner functions and the generic Word Problem for finitely generated amenable groups, J. Algebra, 511 (2018) 388 - 404

Paradoxical decomposition

A **paradoxical decomposition** of a group G is a triple $(K,(A_k)_{k\in K},(B_k)_{k\in K})$ consisting families $\mathcal{A}$ and $\mathcal{B}$ of subsets of G indexed by elements of a finite set $K \subset G$ such that[2]:

$$G = \Big(\bigsqcup_{k\in K} kA_k\Big) \bigsqcup \Big(\bigsqcup_{k\in K} kB_k\Big) = \Big(\bigsqcup_{k\in K} A_k\Big) = \Big(\bigsqcup_{k\in K} B_k\Big).$$

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Definition. When (G, ν) is a computable group and the families $\mathcal{A}$ and $\mathcal{B}$ consist of computable sets, then such a paradoxical decomposition is called **effective**.

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Tarski-Følner Theorem

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Theorem

Let G be a group. The following conditions are equivalent:

1. *G is not amenable;*
2. *G does not satisfy Følner's condition;*
3. *G admits a paradoxical decomposition.*

Effective version of Tarski-Følner Theorem

Theorem

Let G be a computable group. For any $K' \subset G$ that contradicts Følner condition (i.e. for some natural n there is no n-Følner set with respect to K') we can effectively find a finite subset $K \supseteq K'$ such that there is an effective paradoxical decomposition of G of the form $(K, (A_k)_{k \in K}, (B_k)_{k \in K})$.

Notation

Let $\Gamma = (V, E)$ be a graph.

For $X \subset V$ let $N(X) = \{v \in V : \exists x \in X \ (x, v) \in E\}$.

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We denote such a bipartite graph by $\Gamma = (A, B, E)$.

$(1, k)$-matchings

Definition. A **perfect** $(1, k)$-**matching** from A to B is a set $M \subset E$ satisfying following conditions:

1. for every $a \in A$ there exists exactly k vertices $b_1, \ldots b_k \in B$ such that $(a, b_1), \ldots, (a, b_k) \in M$;
2. for every $b \in B$ there is an unique vertex $a \in A$ such that $(a, b) \in M$.

The Hall's Harem Theorem

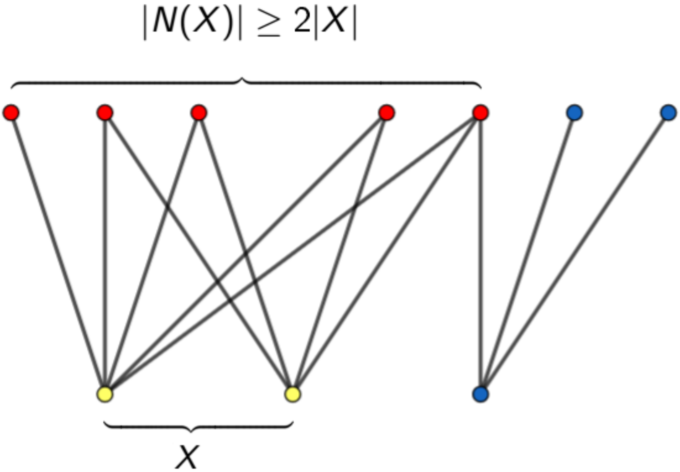
Theorem

Let $\Gamma = (A, B, E)$ be a locally finite graph and let $k \in \mathbb{N}$, $k \geq 1$. The following conditions are equivalent:

1. *For all finite subsets $X \subset A$, $Y \subset B$ following inequalities holds* $|N(X)| \geq k|X|$, $|N(Y)| \geq \frac{1}{k}|Y|$.
2. *Γ has a perfect $(1, k)$-matching.*

Example

Let $k = 2$:



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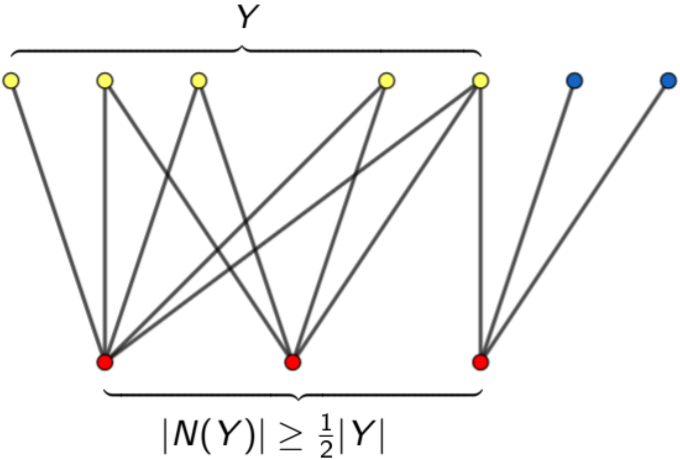

Computable Graphs

Definition. A graph Γ is **computable** if there exists a bijective function $\nu : \mathbb{N} \to V$ such that

$$R := \{(i,j) : (\nu(i), \nu(j)) \in E\}$$

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Definition. A bipartite graph $\Gamma = (A, B, E)$ is **computably bipartite** if Γ is computable and the set of ν-numbers of A is computable.

Computable graphs

Definition. A locally finite graph Γ is called **highly computable**[3] if it is computable and there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) = |N(\nu(n))|$ for all $n \in \mathbb{N}$.

[3] H. Kierstead, An effective version of Hall's Theorem, Proc. Amer. Math. Soc. 88 (1983) 124 - 128

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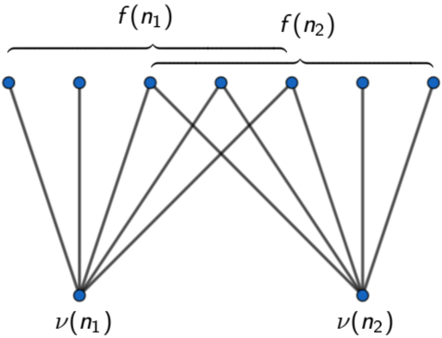

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Computable $(1, k)$-matchings

Definition. Let $\Gamma = (A, B, E)$ be a computably bipartite graph. A perfect $(1, k)$-matching M from A to B is called a **computable perfect** $(1, k)$-**matching** if there is an algorithm which

- for each i with $\nu(i) \in A$, finds the tuple $(i_1, i_2, \ldots, i_k)$ such that $(\nu(i), \nu(i_j)) \in M$, for all $j = 1, 2, \ldots, k$
- when $\nu(i) \notin A$ it finds i' such that $(\nu(i'), \nu(i)) \in M$.

Effective version of Hall's condition

A bipartite graph $\Gamma = (A, B, E)$ satisfies the **computable expanding Hall's harem condition with respect to** k (denoted $c.e.H.h.c.(k)$), if and only if there is a computable function $h : \mathbb{N} \to \mathbb{N}$ such that:

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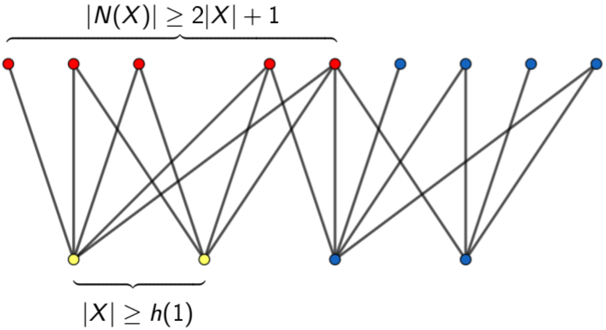
- for all finite sets $X \subset A$, the inequality $h(n) \leq |X|$ implies $n \leq |N(X)| - k|X|$
- for all finite sets $Y \subset B$, the inequality $h(n) \leq |Y|$ implies $n \leq |N(Y)| - \frac{1}{k}|Y|$.

Let $k = 2,\ h(1) = 2$.



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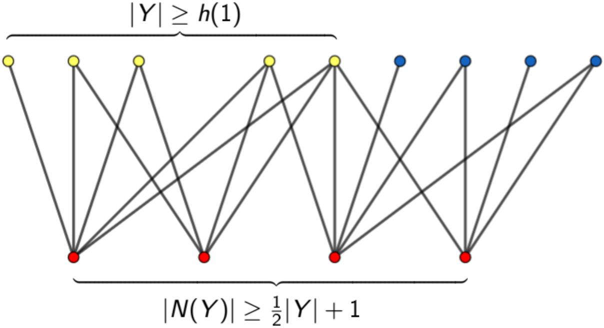

An effective version of Hall's Harem Theorem

Theorem

If $\Gamma = (A, B, E)$ is a highly computable bipartite graph satisfying the c.e.H.h.c.(k), then Γ has a computable perfect $(1, k)$-matching.

Witnesses of the Banach-Tarski paradox

Definition. Let

$$\mathfrak{W}_{BT} = \left\{ K : (K \subset\subset G) \wedge \exists n \in \mathbb{N} \, (\forall F \subset\subset G)(\exists k \in K) \left(\frac{|F \setminus kF|}{|F|} \geq \frac{1}{n} \right) \right\}$$

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We call this family **witnesses of the Banach-Tarski paradox**.

Lemma

Let G be a group, $x, y \in G$ and $\langle x, y \rangle$ be a non-abelian free subgroup of G. Then $\{x, y\} \in \mathfrak{W}_{BT}$.

Definition. Group G is called **fully residually free** if for any finite collection of nontrivial elements $g_1, \ldots, g_n \in G \setminus \{1\}$ there exists a homomorphism $\phi : G \to \mathbb{F}$ onto a free group $\mathbb{F}$ such that $\phi(g_1) \neq 1, \ldots, \phi(g_n) \neq 1$, [4].

[4] I. Kapovich, Subgroup properties of fully residually free groups, Trans. Amer. Math. Soc. 354 (2001) 335 - 362

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Theorem

The family $\mathfrak{W}_{BT}$ is computable for any computable fully residually free group.

[4] I. Kapovich, Subgroup properties of fully residually free groups, Trans. Amer. Math. Soc. 354 (2001) 335 - 362

K. Duda, with an appendix written by A. Ivanov, Amenability and Computability, arXiv:1904.02640.

Thank you for your attention.