Branch structures of some GGS-groups

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GROUPS AND THEIR ACTIONS 2019

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General Burnside Problem

In 1902 William Burnside asked the following question:

"Is a finitely generated periodic group necessarily finite?"

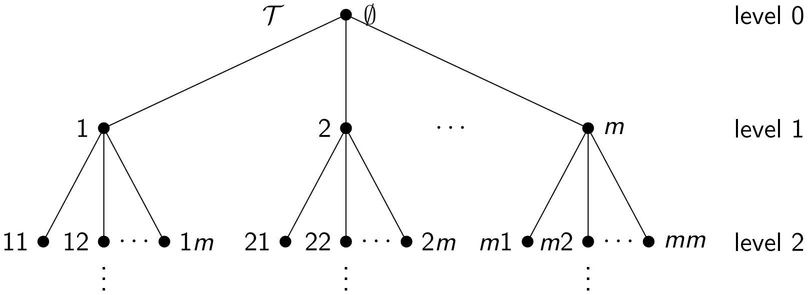
In 1964 Golod and Shafarevich gave examples of infinite p-groups that are finitely generated.

Other counter-examples to the General Burnside Problem:

- the first Grigorchuk group (1980);
- the Gupta-Sidki p-groups (1983)

The regular rooted tree

For an integer $m \geq 2$, the *regular rooted tree* $\mathcal{T}$ of degree m, or simply the *m-adic tree* $\mathcal{T}$, is constructed as follows:

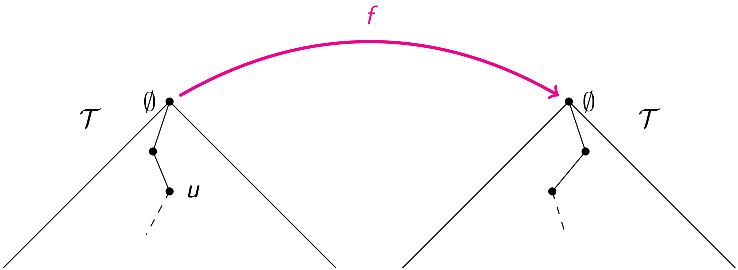


A vertex v in the level n of $\mathcal{T}$ is a word of length n in the alphabet $X = \{1, \dots, m\}$.

Automorphisms of rooted trees

Definition

An automorphism of $\mathcal{T}$ is a bijective map of the set of vertices of $\mathcal{T}$ that preserves incidence.



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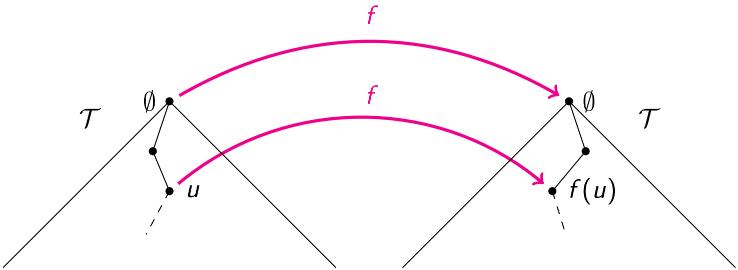

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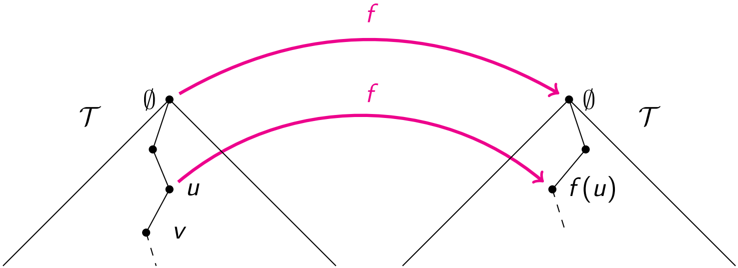

The automorphisms of $\mathcal{T}$ form a group $\operatorname{Aut}\mathcal{T}$ with respect to composition.

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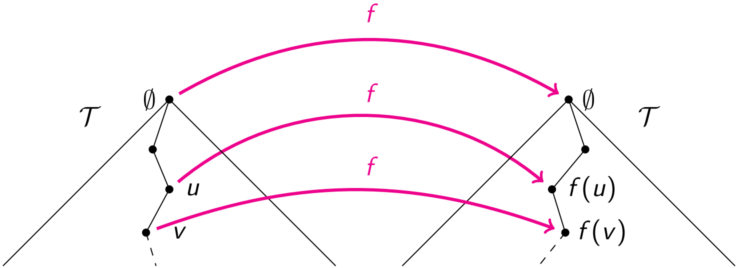
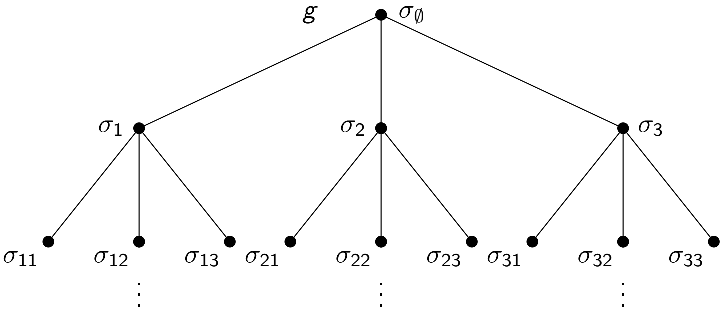


The automorphisms of $\mathcal{T}$ form a group $\mathrm{Aut}\,\mathcal{T}$ with respect to composition.

Describing an automorphism: the portrait

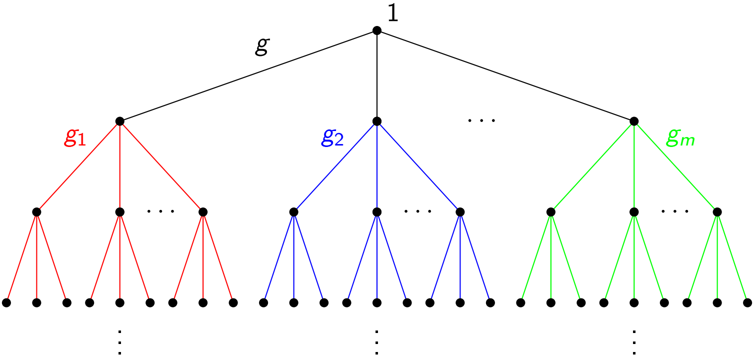
An automorphism $g \in \operatorname{Aut} \mathcal{T}_m$ can be represented by writing in each vertex u a permutation $\sigma_u \in \operatorname{Sym}(m)$ that represents the action of g on the descendants of u. These permutations are called *labels* and the set of all labels is called *the portrait* of g.



Representing an automorphism: the sections

If $g \in \mathrm{Stab}(1)$ we can identify g with a vector whose components are the sections of g hanging from the vertices of the first level.

$$g \longleftrightarrow (g_1, g_2, \dots, g_m)$$

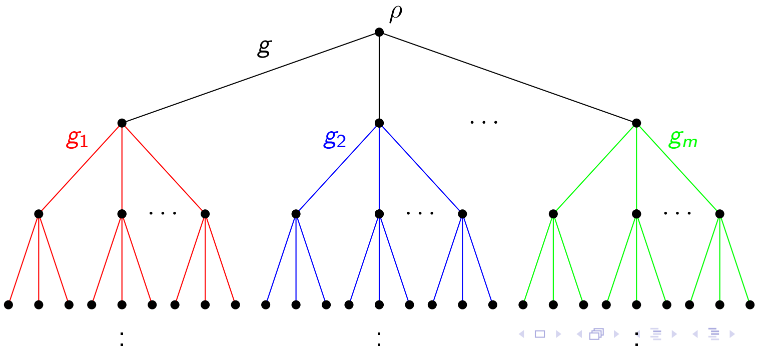


Representing an automorphism: the sections

In the general case an automorphism $g \in \operatorname{Aut} \mathcal{T}_m$ can be written as

$$g = (g_1, \dots, g_m)\rho$$

where ρ is a permutation in $\operatorname{Sym}(m)$ representing the action of g on the first level of $\mathcal{T}$ and $g_1, \dots, g_m$ represent the action of g on the subtrees $\mathcal{T}_i$ hanging from the vertices of the first level.



The GGS-groups

The Grigorchuk-Gupta-Sidki groups, GGS-groups for short, are a generalization of the second Grigorchuk group and the Gupta-Sidki groups. They are two-generated groups acting on an m-adic tree, whose generators are defined by

$$a = (1, \dots, 1)\sigma$$

and

$$b = (a^{e_1}, a^{e_2}, \dots, a^{e_{m-1}}, b)$$

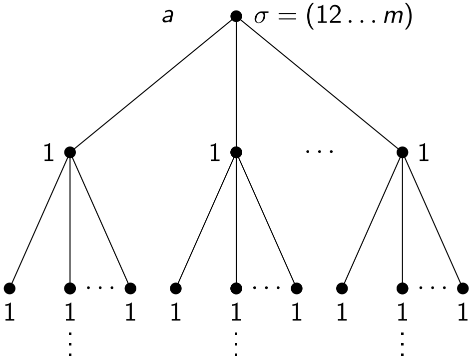
where $\sigma = (1\,2\dots m)$ and $\mathbf{e} = (e_1, \dots, e_{m-1}) \in (\mathbb{Z}/m\mathbb{Z})^{m-1}$ is called the defining vector of the group.

Definition

The group $G = \langle a, b \rangle$ is called the GGS-group corresponding to the defining vector $\mathbf{e}$.

GGS-groups: portrait of a

$$a = (1, \dots, 1)\sigma$$



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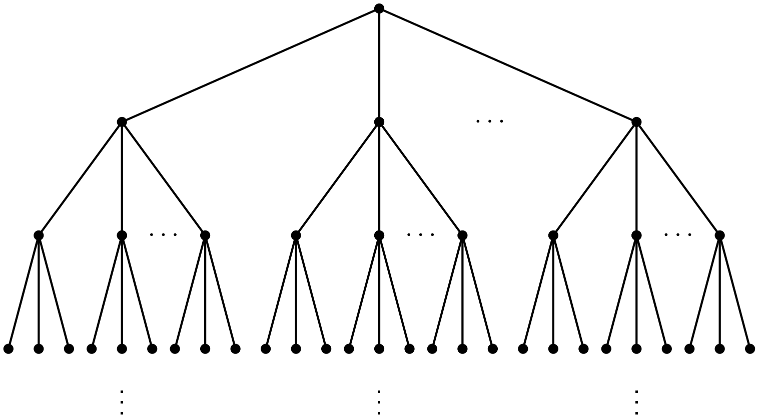

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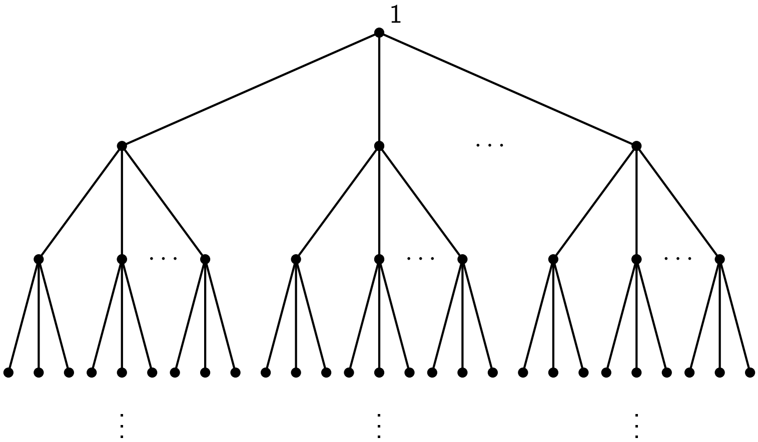

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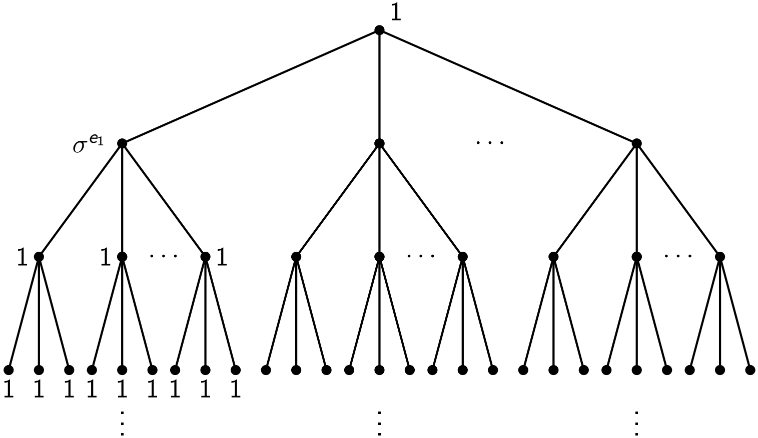

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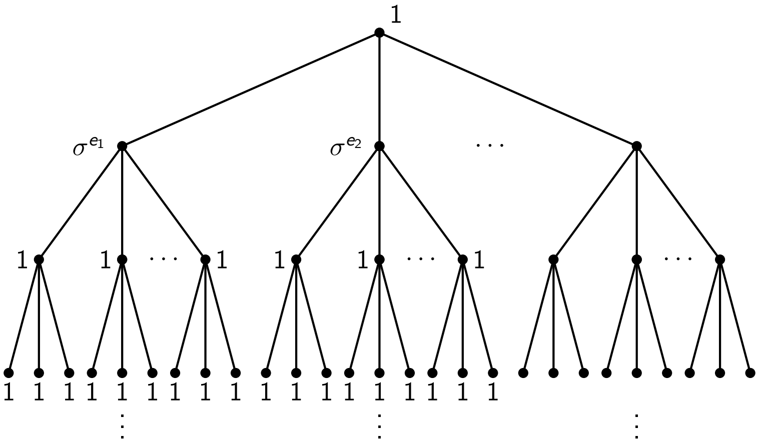

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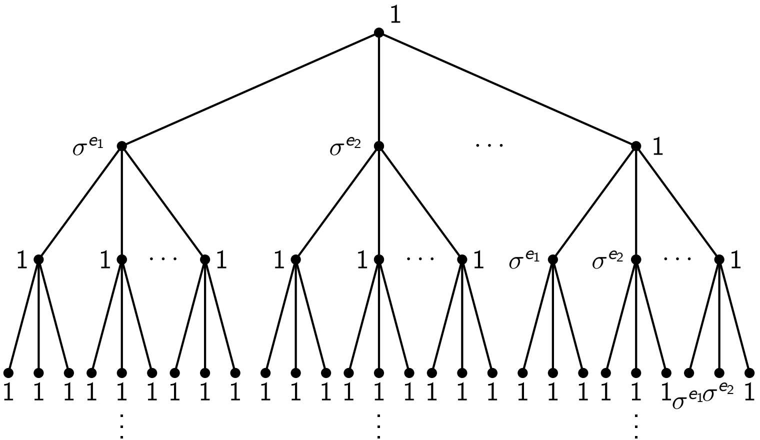

GGS-groups: counterexamples to the General Burnside Problem

Theorem (Vovkivsky)

Let G be the GGS-group corresponding to the defining vector $\mathbf{e} = (e_1, \ldots, e_{p^n-1}) \in (\mathbb{Z}/p^n\mathbb{Z})^{p^n-1}$. Then

(i) *G is infinite if and only if there exists i such that*

$$R_i = R_{i+1} = \cdots < n. \quad (1)$$

(ii) *G is a periodic group if and only if for each $k = 0, \ldots, n-1$*

$$S_k \equiv 0 \pmod{p^{k+1}} \quad (2)$$

Where $S_k = e_{p^k} + e_{2p^k} + \cdots + e_{p^n-p^k}$.

Thus conditions (4) and (5) together give a criterion for the group G to be an infinite p-group.

A special property of Aut $\mathcal{T}$

If $g \in \mathrm{Stab}(1)$ we can identify g with a vector whose components are the sections of g hanging from the vertices of the first level.

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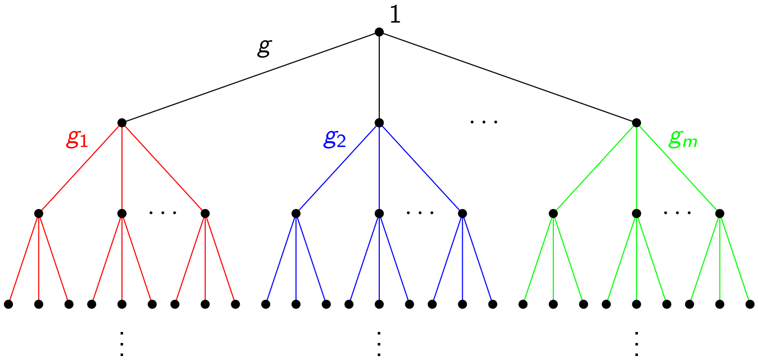

A special property of Aut $\mathcal{T}$

We have the following isomorphism:

$$\begin{array}{rccc} \psi: & \mathrm{Stab}(1) & \rightarrow & \mathrm{Aut}\,\mathcal{T} \times \overset{m}{\cdots} \times \mathrm{Aut}\,\mathcal{T} \\ & g & \rightarrow & (g_1, \ldots, g_m) \end{array}$$

This isomorphism means that inside Aut $\mathcal{T}$ we can find the direct product of Aut $\mathcal{T}$ as many times as we want

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Regular branch groups

Definition

A subgroup G of $\mathrm{Aut}\,\mathcal{T}$ is *spherically transitive* if it acts transitively on each level of $\mathcal{T}$.

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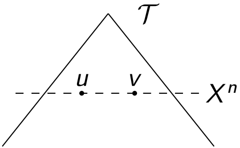
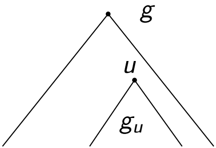
A subgroup G of $\mathrm{Aut}\,\mathcal{T}$ is *self-similar* if for all $u \in X^*$ the section $g_u \in G$.



Spherically transitive:

$$\forall u, v \in X^n \; \exists \; g \in G \mid g(u) = v$$



Self-similar

$$\forall u \in X^n \text{ and } \forall g \in G \;\; g_u \in G$$

Regular branch groups

$$\begin{array}{rccc} \psi : & \mathrm{Stab}(1) & \rightarrow & \mathrm{Aut}\,\mathcal{T} \times \overset{m}{\cdots} \times \mathrm{Aut}\,\mathcal{T} \\ & g & \rightarrow & (g_1, \ldots, g_m) \end{array}$$

Definition

Let G be a self-similar spherically transitive group of automorphisms of the m-adic tree $\mathcal{T}$, and let K be a non-trivial subgroup of $\mathrm{Stab}_G(1)$. We say that G is *weakly regular branch over* K if

$$K \times \overset{m}{\cdots} \times K \subseteq \psi(K). \tag{3}$$

If furthermore K has finite index in G, we say that G is *regular branch over* K.

A criterion for a GGS-group to be a branch group

Theorem (Vovkivsky)

Let G be the GGS-group corresponding to the defining vector $\mathbf{e} = (e_1, \dots, e_{p^n-1}) \in (\mathbb{Z}/p^n\mathbb{Z})^{p^n-1}$. If G is an infinite and periodic group then it is a regular branch group if and only if there exists $k \in \{1, \dots, p^n - 1\}$ such that $e_k \not\equiv 0 \mod p$.

Actually he proved that

$$G'' \times \overset{p^n}{\cdots} \times G'' \subseteq \psi(G'')$$

so in this case G is regular branch over G''.

GGS-groups: counterexamples to the General Burnside Problem

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(i) *G is infinite if and only if there exists i such that*

$$R_i = R_{i+1} = \cdots < n. \quad (4)$$

(ii) *G is a periodic group if and only if for each $k = 0, \ldots, n-1$*

$$S_k \equiv 0 \pmod{p^{k+1}} \quad (5)$$

Where $S_k = e_{p^k} + e_{2p^k} + \cdots + e_{p^n-p^k}$.

Thus conditions (4) and (5) together give a criterion for the group G to be an infinite p-group.

Branch structure of some non-periodic GGS-groups over p^n-adic trees

Theorem (E.D., N. Gavioli)

Let G be a GGS-group acting on $\mathcal{T}_{p^n}$ with defining vector $\mathbf{e} = (e_1, \ldots, e_{p^n-1})$. Let e_k be the first component of $\mathbf{e}$ such that $e_k \not\equiv 0 \pmod{p}$. If

$$e_{p^n-k} \equiv 0 \pmod{p^n}$$

then G is a regular branch group over G'

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$$\begin{vmatrix} e_{m-k} & e_m \\ e_m & e_{m+k} \end{vmatrix} \not\equiv 0 \pmod p$$

then G is a regular branch over $\gamma_3(G)$.

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$$\begin{vmatrix} e_{m-k} & e_m \\ e_m & e_{m+k} \end{vmatrix} \equiv 0 \pmod{p}.$$

If one of the following conditions holds

(i) $e_{2k} \not\equiv e_k \pmod{p}$

(ii) $k \nmid p^n$

then G is a regular branch group.

Branch structure of some non-periodic GGS-groups over p^n-adic trees

Theorem (E.D., N. Gavioli)

Let G be a GGS-group over the p^n-adic tree. Then G is weakly branch and, furthermore, if the defining vector does not belong to $\mathcal{E}$ then G is regular branch over either G' or $\gamma_3(G)$.

Where

$$\mathcal{E} = \bigcup_{t=0}^{n-1} \mathcal{E}_t$$

and for $t \in \{0, 1, \dots, n-1\}$ the set $\mathcal{E}_t$ is defined as the set of the defining vector $\mathbf{e} \in (\mathbb{Z}/p^n\mathbb{Z})^{p^n-1}$ such that

$$\begin{cases} e_{p^t} \equiv e_{p^{2t}} \equiv \cdots \equiv e_{p^n-p^t} \not\equiv 0 \mod p \\ e_i \equiv 0 \mod p & \textit{otherwise} \end{cases}$$

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- G. A. Fernández-Alcober and Amaia Zugadi-Reizabal. *GGS-groups: order of congruence quotients and Hausdorff dimension*. Trans. Amer. Math. Soc. 366 (2014), n. 4, 1993–2017.
- G. A. Fernández-Alcober, A. Garrido, and J Uria-Albizuri *On the congruence subgroup property for GGSgroups* Proc. Amer. Math. Soc. 145 (2017), no. 8, 3311-3322. MR 3652785

Thank you!