On the generic family of Cayley graphs of a finite group

Czesław Bagiński, Piotr Grzeszczuk

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CB, Piotr Grzeszczuk

On the generic family of Cayley graphs of a finite group

Cayley graph

Definition

Let G be a group, e - its unity. A subset $S \subset G \setminus \{e\}$ is *symmetric*, if

$$S = S^{-1},$$

where $S^{-1} = \{s^{-1} | \, s \in S\}$.

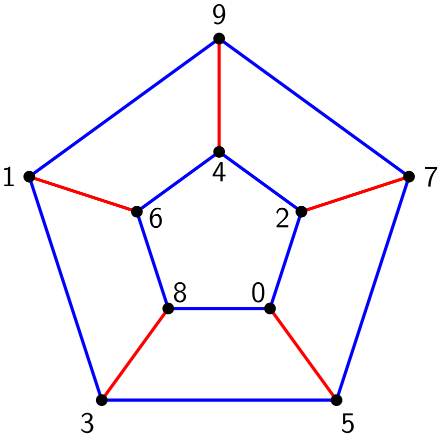
The Cayley graph of G with respect to S,

$$Cay(G, S)$$

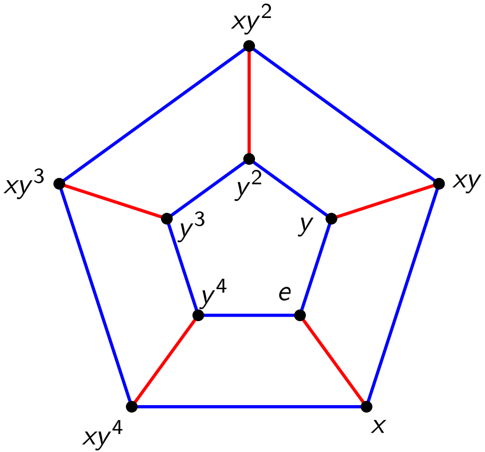
is a graph, whose set of vertices is G and the set of edges is defined by the condition:

$$g \sim h \;\Leftrightarrow\; \exists_{s \in S} \; h = s \cdot g.$$

Cayley graph



$$G = \mathbb{Z}_{10} = \{0,1,2,3,4,5,6,7,8,9\}$$
$$S = \{5,\ 2,\ 8\}$$



$$G = D_5 = \langle x,\ y :\ x^2 = e = y^5,\ xyx = y^{-1} \rangle$$

$$S = \{x,\ y,\ y^{-1}\}$$

Definition of the graph $\mathscr{G}_m(G)$

Definition

Let G - a finite group and $m \in \mathbb{N}$. *The generic graph of G of degree m*, is the Cayley graph

$$\mathscr{G}_m(G) = Cay(G^m, S),$$

where

$$G^m = \underbrace{G \times G \times \cdots \times G}_{m},$$

$$S = \left\{ \mathbf{x}_{[k,l)} : \ x \in G^{\times}, 1 \leqslant k < l \leqslant m+1 \right\}$$

$$\mathbf{x}_{[k,l)} = (\underbrace{e, e, \ldots, e}_{k-1 \ \text{times}}, \underbrace{x, x, \ldots, x}_{l-k \ \text{times}}, e, e, \ldots, e).$$

Definition of the graph $\mathscr{G}_m(G)$

Definition

An interval of G^m is the set

$$G_{[k,l)} = \{\mathbf{x}_{[k,l)} : x \in G \setminus \{e\}\}.$$

Hence

$$S = \bigcup_{1 \leqslant k < l \leqslant m+1} G_{[k,l)}.$$

Therefore for vertices $\mathbf{g} = (g_1, \ldots, g_m)$, $\mathbf{h} = (h_1, \ldots, h_m)$ of $\mathscr{G}_m(G)$,

$$\mathbf{g} \sim \mathbf{h} \Leftrightarrow \exists_{x \in G^\times}\ \mathbf{h} = \mathbf{x}_{[k,l)} \cdot \mathbf{g} \text{ for some } 1 \leqslant k < l \leqslant m+1.$$

Lemma 1.

(a) *The graph* $\mathscr{G}_m(G)$ *has* $|G|^m$ *vertices.*

(b) *Every vertex of the graph* $\mathscr{G}_m(G)$ *has degree*

$$d = |S| = \binom{m+1}{2}(|G| - 1).$$

Definition of the graph $\mathscr{G}_m(G)$

Lemma 1.

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Example

m	$\binom{m+1}{2}$	$\lvert G\rvert$	$\lvert V\rvert$	$d = S$
2	3	60	3600	177
3	6	60	216 000	354
6	21	6	46 656	105

Motivation

Suppose, that a finite group G acts on a non-commutative ring R by automorphisms

$$G \to \mathbf{Aut}(R), \quad r \mapsto r^g.$$

The subring of fixed points

$$R^G = \{r \in R \mid r^g = r \ \text{ for all } \ g \in G\}.$$

A natural way to construct fixed points of the action is to use the trace map $\mathbf{tr}_G \colon R \to R$ defined by

$$\mathbf{tr}_G(r) = \sum_{g \in G} r^g.$$

The image

$$T = \mathbf{tr}_G(R)$$

is an ideal of R^G.

Motivation

Theorem (Bergman - Isaacs)

Let G be a finite group of automorphisms of the ring R with no additive $|G|$-torsion. If $T = \mathbf{tr}_G(R)$ is nilpotent of index d, then R is nilpotent of index at most $f(|G|)^d$, where

$$f(m) = \prod_{k=1}^{m} \left(\binom{m}{k} + 1 \right).$$

In particular, if $\mathbf{tr}_G(R) = 0$, then

$$R^{f(|G|)} = 0.$$

Function f

$$f(1) = 2,\ \ f(3) = 6,\ \ f(10) = 84\,447\,578\,671\,097\,576$$

Function f

$$f(1) = 2, \;\; f(3) = 6, \;\; f(10) = 84\,447\,578\,671\,097\,576$$

Conjecture

Under assumtions of the Bergman-Isaacs theorem the nilpotency index of R is not bigger than $|G|^d$.
(It was proved for solvable groups.)

$f(60) =$

Motywacje

$f(60) =$

10027035042685832144316239238423228478222451493397161499
36464455677719685916368387636892763452418041134076338288
69862998830877422902252855651665353729324393624179813297
93559877203994959612279078824215598138834690984524914567
38527924253254908608597225201038183332273908723660430932
08558345176356925340234940682039681247438926960443750994
34719821515897802677353110036988721221819637408341376652
48048420308579120482903924113949173687308827596832676712
78947772358216607226376849553847036033685649495999722401
57268054752900810875459238960361413218191340520754827322
54690341912167129059327087241364234782914340849047237217
54991535454822020690513792920584033071200626006852980210
975690367331745697642603152835712254726210436389561958400

Motivation

For a given group G and a set I let

$$Z_G = \mathbb{Z}\langle \zeta_{i,g} \mid g \in G, i \in I\rangle \quad \text{and} \quad Q_G = \mathbb{Q}\langle \zeta_{i,g} \mid g \in G, i \in I\rangle$$

be algebras without 1 over $\mathbb{Z}$ and $\mathbb{Q}$ respectively.
The group G acts on both algebras:

$$(\zeta_{i,g})^x = \zeta_{i,x^{-1}g}.$$

For an arbitrary ring A, acted upon by G, one can take a sufficiently large set I, such that there exists a G-epimorphism θ of Z_G onto A with

$$\theta(\mathbf{tr}_G(Z_G)) = \mathbf{tr}_G(A).$$

This allows to reduce the investigation of Conjecture to combinatorial properties of the graph $\mathscr{G}_m(G)$.

Motivations

Proposition 2.

Let A be the adjacency matrix of $\mathscr{G}_m(G)$. If for every eigenvalue λ of A

$$\lambda > -\binom{m+1}{2},$$

then, under the assumptions of the Bergman-Isaacs theorem, the nilpotency index of R is not bigger than m.

Proposition 3.

Let G be a finite abelian group. If $m \geqslant |G|$, then every eigenvalue λ of A satisfies the inequality

$$\lambda > -\binom{m+1}{2}$$

In particular the nilpotency index of R is not bigger than m.

Theorem 4.

If $\mathcal{G} = Cay(G, S)$ is a Cayley graph, then the set $\mathbf{T}$ *of functions of the form*

$$T_g : G \to G, \quad x^{T_g} = xg$$

is a vertex transitive group of automorphisms of $\mathcal{G}$ (isomorphic with G).

Corollary.

For every $g \in G$ the subgraph $\mathcal{V}(g)$ of $\mathscr{G}_m(G)$, whose set of vertices is equal to the set of all neighbours of g is isomorphic to $\mathcal{V}(e)$.

Example

Let a, b, c be different elements of $G^\times$ and let

$$\mathbf{g} = (e, a, a, b, b, b, c, e, e), \ \mathbf{h} = (e, a, e, e, b, b, c, c, e).$$

Then they have the following decompositions, which we call *weight decompositions*; the number of components we call *the weight*:

$$\mathbf{g} = \mathbf{a}_{[2,4)}\mathbf{b}_{[4,7)}\mathbf{c}_{[7,8)} \text{ and } \vartheta(\mathbf{g}) = 3$$

$$\mathbf{h} = \mathbf{a}_{[2,3)}\mathbf{e}_{[3,5)}\mathbf{b}_{[5,7)}\mathbf{c}_{[7,9)} \text{ and } \vartheta(\mathbf{h}) = 4.$$

Lemma 5.

Let G be a group and $m > 1$. For every $\mathbf{g} \in G^m \setminus \{\mathbf{e}\}$ there exist a sequence

$$1 \leqslant i_1 < i_2 < \cdots < i_k < i_{k+1} \leqslant m+1$$

and elements

$$x_1, \ldots, x_k \in G,$$

such that

1. $x_1 \neq e \neq x_k$;
2. $x_i \neq x_{i+1}$ *for* $i = 1, 2, \ldots, k-1$;
3. $\mathbf{g} = \mathbf{x}_{1[i_1,i_2)}\mathbf{x}_{2[i_2,i_3)} \cdots \mathbf{x}_{k[i_k,i_{k+1})}$.

The presentation of $\mathbf{g}$ in this form is unique.

Definition.

The number $\vartheta(\mathbf{g}) = k$ we call *the weight* of an element $\mathbf{g}$. It is clear that $1 \leqslant \vartheta(\mathbf{g}) \leqslant m$ for every $\mathbf{g} \neq \mathbf{e}$.

Lemma 6.

If $\mathbf{g}, \mathbf{h} \in G^m \setminus \{\mathbf{e}\}$ *are adjacent vertices in* $\mathscr{G}_m(G)$*, then* $|\vartheta(\mathbf{g}) - \vartheta(\mathbf{h})| \leqslant 2$.

For $\mathbf{g} \in G^m$ let $V(\mathbf{g})$ be the set of all neighbours of $\mathbf{g}$ in $\mathscr{G}_m(G)$.

Proposition 7.

For everey element $\mathbf{g} \in G^m \setminus \{\mathbf{e}\}$:

(a) $\mathbf{g} \in V(\mathbf{e})$ *iff* $\vartheta(g) = 1$. *In this case we have* $|V(\mathbf{e}) \cap V(\mathbf{g})| = |G| + 2m - 4$;

(b) *if* $\vartheta(\mathbf{g}) = 2$, *then* $|V(\mathbf{e}) \cap V(\mathbf{g})| = 6$;

(c) *if* $\vartheta(\mathbf{g}) = 3$, *then* $|V(\mathbf{e}) \cap V(\mathbf{g})|$ *belongs to* $\{0, 1, 2, 4, 6\}$;

(d) *if* $\vartheta(\mathbf{g}) \geqslant 4$, *then* $V(\mathbf{e}) \cap V(\mathbf{g}) = \emptyset$.

Corollary 8.

A group G is nonabelian if and only if for $m \geqslant 3$ there exist two vertices in $\mathscr{G}_m(G)$ which have exactly one common neighbour.

Definition

A k-regular graph X on n vertices is called

edge regular

if there exists a parameter a such that every two adjcent vertices have exactly a common neighbours. In this case we say that X is edge regular with the parameters

$$(n, k, a).$$

Corollary 9.

The graph $\mathscr{G}_m(G)$ is edge regular r with parameters:

$$(|G|^m,\ \binom{m+1}{2}(|G|-1),\ |G|+2m-4).$$

Definition

A graph X is called

strongly regular

with parameters (n, k, a, c) when it is edge regular with parameters (n, k, a) and every two of non-adjacent vertices have c common neighbours.

Corollary 10.

(a) The graph $\mathscr{G}_m(G)$ is strongly regular if and only if $m = 2$.
(b) For a given group G the graph $\mathscr{G}_2(G)$ is strongly regular with parameters

$$(|G|^2, 3(|G| - 1), |G|, 6).$$

The eigenvalues of the adjacency matrix of $\mathscr{G}_2(G)$ are equal

$$3(|G| - 1),\ \ |G| - 3,\ \ -3$$

with multiplicities

$$1,\ \ 3(|G| - 1),\ \ |G|^2 - 3|G| + 2$$

respectively.

Definition

By

$$\mathscr{V}_m(g) = \mathscr{V}_m(g, G)$$

we denote a subgraph of $\mathscr{G}_m(G)$ whose set of vertices is equal to $V(g)$.

For an element $x \in G^{\times}$ by

$$\mathscr{I}_m(x) = \mathscr{I}_m(x, G)$$

we denote a subgraph of $\mathscr{V}_m(e, G)$, whose set of vertices is equal

$$\{\mathbf{x}_{[k,l)} :\ 1 \leq k < l \leq m+1\} \cup \{\mathbf{x}_{[k,l)}^{-1} :\ 1 \leq k < l \leq m+1\}.$$

Lemat 11.

(a) The number of vertices of $\mathscr{I}_m(x)$ is equal to

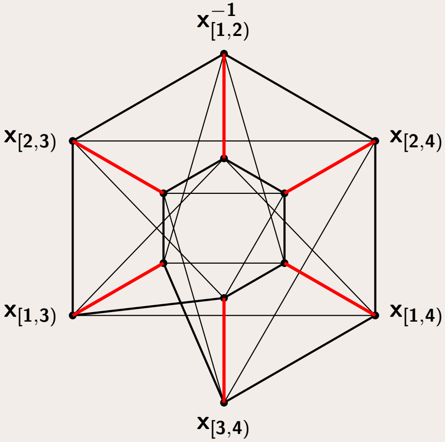
$$\binom{m+1}{2}$$

when x has order 2 and twice more, when x has order bigger than 2.

(b) For a fixed interval $[k,l)$, in the graph $\mathscr{I}_m(x)$ the vertex $\mathbf{x}_{[k,l)}$ is adjacent to

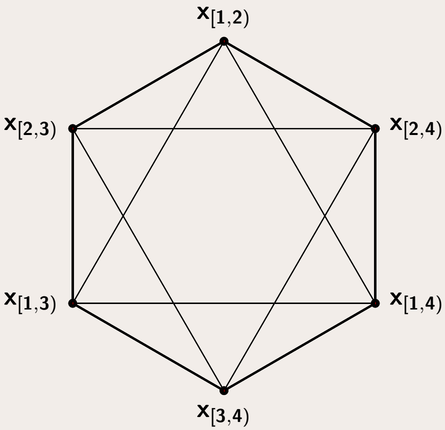
- *$m-k+l-2$ vertices of the form $\mathbf{x}_{[i,j)}$ (where $k=i$, $k<j$ and $l\neq j$ or $l=j$, $i<l$ and $k\neq i$),*
- *$m+k-l$ vertices of the form $\mathbf{x}^{-1}_{[i,j)}$ (where $k=j+1$ and $i<k$ or $l=i$ and $l<j$),*
- *$\mathbf{x}^{-1}_{[k,l)}$ (for $o(x)>2$ only).*

Elementary properties of the graph $\mathscr{G}_m(G)$

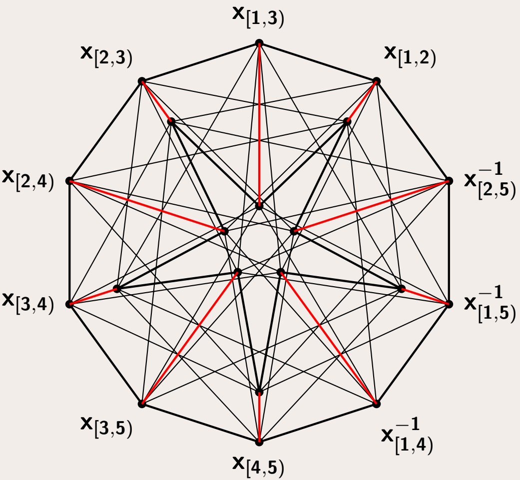


Graph $\mathscr{I}_3(x)$, $o(x) > 2$

Elementary properties of the graph $\mathscr{G}_m(G)$

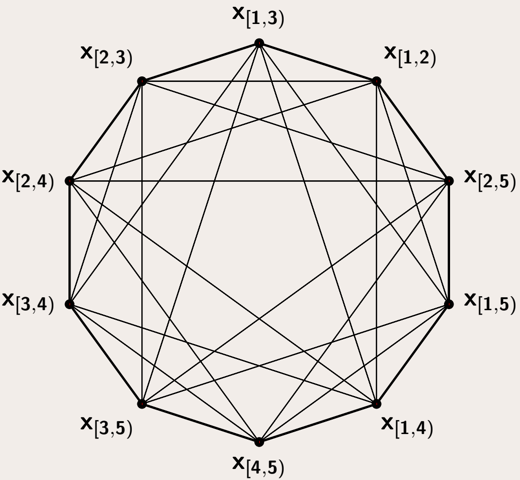


Graph $\mathscr{I}_3(x)$, $o(x) = 2$



Graph $\mathscr{I}_4(x)$, $o(x) > 2$

Elementary properties of the graph $\mathscr{G}_m(G)$



Graph $\mathscr{I}_4(x), \quad o(x) = 2$

Elementary properties of the graph $\mathscr{G}_m(G)$

Lemma 12.

Let $m > 1$. For nontrivial groups G and H end elements $x \in G^\times$ and $y \in H^\times$

$$\mathscr{I}_m(x, G) \simeq \mathscr{I}_m(y, H) \ \text{ iff } \ o(x) = 2 = o(y) \ \text{ or } \ o(x) \neq 2 \neq o(y).$$

Proposition 13.

Let G and H be groups such that $|G| = |H|$. If G and H have the same numbers of elements of order 2, then for all $g \in G$ and $h \in H$

$$\mathscr{V}_m(g, G) \simeq \mathscr{V}_m(h, H).$$

Definition

Let $\mathscr{B}_m$ be a graph whose vertices are intervals

$$G_{[k,l)},$$

$1 \leqslant k < l \leqslant m+1$. Two intervals are adjacent

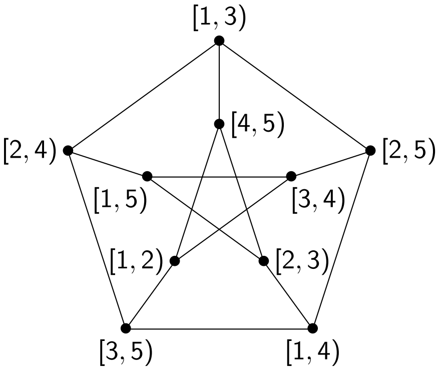
$$G_{[k,l)} \sim G_{[i,j)},$$

if there exist $\mathbf{x}_{[k,l)} \in G_{[k,l)}$ and $\mathbf{y}_{[i,j)} \in G_{[i,j)}$ which are adjacent in $\mathscr{G}_m(G)$.

Note that $\mathscr{B}_m$ is isomorphic to the graph $\mathscr{I}_m(x, G)$, where x is an element of order 2. $\mathscr{B}_m$ can be obtained also from $\mathscr{I}_m(x, G)$, where x has order bigger than 2, by removing the edges $\mathbf{x}_{[k,l)} \sim \mathbf{x}_{[k,l)}^{-1}$, with simultaneous merging of ends and identifying suitable edges to avoid multiple edges.

Proposition 14.

The complement $\overline{\mathscr{B}}_m$ of the graph $\mathscr{B}_m$ is isomorphic to the Kneser graph $KG_{m+1,2}$.



$m = 4, \quad \text{graf } \overline{\mathcal{B}}_4 = KG_{5,2}$

Proposition 15.

The maximal number of vertices in a clique of the graph $\mathscr{G}_m(G)$ is equal to

$$\begin{cases} \max\{m+1, |G|)\} & \text{if} \quad m \geqslant 3 \quad \text{or} \quad |G| > 2, \\ 4 & \text{if} \quad m = 2 \quad \text{and} \quad |G| = 2, \end{cases}$$

Proposition 16.

Let $F : \mathscr{G}_m(G) \to \mathscr{G}_m(H)$ be an isomorphism of graphs such that $F(\mathbf{e}_G) = \mathbf{e}_H$, where G and H are finite groups and let $\mathcal{Q}$ be a clique in $\mathscr{G}_m(G)$. Then $F(\mathcal{Q})$ is a maximum interval (resp. maximum dispersed) clique in $\mathscr{G}_m(H)$ if and only if $\mathcal{Q}$ is a maximum interval (resp. maximum dispersed) clique in $\mathscr{G}_m(G)$, with the exception of the case where $|G| = 3$ and $m = 2$. In particular, if $(m, |G|) \neq (2, 3)$, then any automorphism of the graph $\mathscr{G}_m(G)$ fixing $\mathbf{e}$ preserves the type of a maximum clique.

Homogeneous homomorphisms

Definition

If X and Y are graphs with sets of vertices $V(X)$ and $V(Y)$ respectively, then the function $F\colon V(X) \to V(Y)$ is called *a homomorphism* if

$$x \sim_X y \Rightarrow F(x) \sim_Y F(y).$$

.

Let G i H be groups, with units $\mathbf{e}_G$ and $\mathbf{e}_H$ respectively. A homomorphism of graphs

$$F\colon \mathscr{G}_m(G) \to \mathscr{G}_m(H)$$

is called *homogeneous* if

$$F(\mathbf{e}_G) = \mathbf{e}_H \text{ i } F(G_{[k,l)}) \subseteq H_{[k,l)},$$

for all $1 \leqslant k < l \leqslant m+1$.

Theorem 17.

Let G, H be groups and $m > 1$. Then every homogeneous graph homomorphism (isomorphism) $F\colon \mathscr{G}_m(G) \to \mathscr{G}_m(H)$ is induced by a group monomorphism (isomorphism), that is

$$F(g_1, g_2, \ldots, g_m) = (f(g_1), f(g_2), \ldots, f(g_m))$$

for some monomorphism (isomorphism) of groups $f\colon G \to H$.

Corollary 18.

Let G, H be groups and $m \geqslant 2$. Then the graph $\mathscr{G}_m(G)$ and $\mathscr{G}_m(H)$ are isomorphic with respect to some homogeneous isomorphism iff the groups G and H are isomorphic.

In every Cayley graph right transfers create vertex transitive group of automorphisms of the graph:

$$\mathbf{T}_m(G) = \{T_{\mathbf{g}} : \mathbf{g} \in G^m\}, \;\; T_{\mathbf{g}} : G^m \to G^m, \;\; \mathbf{x}^{T_{\mathbf{g}}} = \mathbf{xg}, \;\; \text{for } \mathbf{x} \in G^m.$$

What is the stabilizer of an arbitrary vertex $(\mathbf{e})$?
The homogeneous automorphisms stabilize $\mathbf{e}$:

$$\mathbf{Aut}_m(G)$$

If $f \in \mathrm{Aut}(G)$ and $\mathbf{x} = (x_1, \ldots, x_m) \in G^m$,

$$\mathbf{x}^f = (f(x_1), \ldots, f(x_m)).$$

It is clear that for every $\mathbf{g} \in G^m$ i $f \in \mathbf{Aut}_m(G)$

$$f^{-1} T_{\mathbf{g}} f = T_{\mathbf{g}^f}$$

which means that $\mathbf{Aut}_m(G)$ normalizes $\mathbf{T}_m(G)$.

Lemma 19.

Let G be an abelian group and $m \geqslant 2$. For $i = 1, 2, \ldots, m$ let $\gamma_i \colon G^m \to G^m$ be the mappings given by

$$(g_1, g_2, \ldots, g_m)^{\gamma_i} = (g_1, \ldots, g_{i-1}, g_{i-1}g_i^{-1}g_{i+1}, g_{i+1}, \ldots, g_m),$$

(we asume $g_0 = g_{m+1} = e$). Then

1. *all γ_i are automorphisms of the group G^m of order 2, satisfying the condition $\mathcal{S}^{\gamma_i} = \mathcal{S}$ and then all they are automorphism of the graph $\mathscr{G}_m(G)$.*
2. *for $|i - j| > 1$, $(1 \leqslant i, j \leqslant m)$, we have $\gamma_i\gamma_j = \gamma_j\gamma_i$.*
3. *for all i, j, $(1 \leqslant i,\ j \leqslant m,\ \ i + j \leqslant m)$, $\gamma_i\gamma_{i+1}\ldots\gamma_{i+j}$ is an automorphism of $\mathscr{G}_m(G)$ of order $j + 2$, in particular the automorphisms $\gamma_i\gamma_{i+1}$ have order 3 and $\gamma_1\ldots\gamma_m = \omega$ has order $m + 1$.*
4. *the subgroup $\mathbf{\Gamma}_m = \langle\gamma_1, \gamma_2, \ldots, \gamma_m\rangle$ of $\mathbf{Aut}(\mathscr{G}_m(G))$ is isomorphic to the symmetric group S_{m+1} of degree $m + 1$.*

Theorem 20.

Let G be an abelian group. If either

(a) $m > 3$, *or*

(b) $m = 3$ *and G is of exponent bigger than* 2, *or*

(c) $m = 2$ *and* $|G| \neq 3$,

then

1. *the stabilizer of* $\mathbf{e} \in G^m$ *in the automorphism group* $\mathbf{Aut}(\mathscr{G}_m(G))$ *is equal to*

$$\mathbf{Aut}_m(G) \times \mathbf{\Gamma}_m \simeq \mathbf{Aut}(G) \times S_{m+1};$$

2. *the group of all automorphisms of the graph* $\mathscr{G}_m(G)$ *is equal to*

$$\mathbf{Aut}(\mathscr{G}_m(G)) = \mathbf{T}_m \rtimes (\mathbf{Aut}_m(G) \times \mathbf{\Gamma}_m) \simeq G^m \rtimes (\mathbf{Aut}(G) \times S_{m+1}).$$

Lemma 21.

Let G be a group and $m \geqslant 2$. Then

1. *the map $\tau\colon G^m \to G^m$ given by*

$$(g_1, g_2, \dots, g_m)^\tau = (g_m, g_{m-1}, \dots, g_2, g_1)$$

is an automorphism of order two of $\mathscr{G}_m(G)$,

2. *The map $\omega\colon G^m \to G^m$ given by*

$$(g_1, g_2, \dots, g_m)^\omega = (g_1^{-1} g_2, g_1^{-1} g_3, \dots, g_1^{-1} g_m, g_1^{-1})$$

is an automorphism of order $m+1$ of $\mathscr{G}_m(G)$,

3. *the subgroup $\mathbf{\Delta}_m = \langle \tau, \omega \rangle$ of $\mathbf{Aut}(\mathscr{G}_m(G))$ is isomorphic to the dihedral group D_{m+1}.*

Similarly as in the abelian case

$$\mathbf{Aut}_m(G) \text{ normalizes } \mathbf{T}_m$$

but for nonabelian groups the automorphism ω z $\mathbf{\Delta}_m(G)$ does not normalize $\mathbf{T}_m$. In fact

$$\begin{aligned}
&(x_1, x_2, \ldots, x_m)^{\omega^{-1} T_{\mathbf{g}} \omega} = \\
&= (x_m^{-1}, x_m^{-1} x_1, x_m^{-1} x_2, \ldots, x_m^{-1} x_{m-2}, x_m^{-1} x_{m-1})^{T_{\mathbf{g}} \omega} \\
&= (x_m^{-1} g_1, x_m^{-1} x_1 g_2, x_m^{-1} x_2 g_3, \ldots, x_m^{-1} x_{m-2} g_{m-1}, x_m^{-1} x_{m-1} g_m)^{\omega} \\
&= (g_1^{-1} x_1 g_2, g_1^{-1} x_2 g_3, g_1^{-1} x_3 g_4, \ldots, g_1^{-1} x_{m-1} g_m, g_1^{-1} x_m) \\
&= (x_1^{g_1} (g_1^{-1} g_2), x_2^{g_1} (g_1^{-1} g_3), x_3^{g_1} (g_1^{-1} g_4), \ldots, x_{m-1}^{g_1} (g_1^{-1} g_m), x_m^{g_1} g_1^{-1}) \\
&= (x_1, x_2, x_3, \ldots, x_{m-1}, x_m)^{f_{g_1} T_{\mathbf{g}^{\omega}}},
\end{aligned}$$

Thus

$$\omega^{-1} T_{\mathbf{g}} \omega = f_{g_1} T_{\mathbf{g}^{\omega}},$$

where f_{g_1} is an internal automorphism induced by $g_1 \in G$.

Theorem 22.

Let G be a non-abelian group. Then

1. the stabilizer of $\mathbf{e} \in G^m$ in the automorphism group $\mathbf{Aut}(\mathscr{G}_m(G))$ is equal to

$$\mathbf{Aut}_m(G) \times \mathbf{\Delta}_m \simeq \mathbf{Aut}(G) \times D_{m+1};$$

2. the group of all automorphisms of the graph $\mathscr{G}_m(G)$ is equal

$$\mathbf{Aut}(\mathscr{G}_m(G)) = \left(\mathbf{T}_m \rtimes \mathbf{Aut}_m(G)\right) \rtimes \mathbf{\Delta}_m \simeq \left(G^m \rtimes \mathbf{Aut}(G)\right) \rtimes D_{m+1}.$$

Theorem 23.

Let G and H be groups and $m > 1$. Then the graphs $\mathscr{G}_m(G)$ and $\mathscr{G}_m(H)$ are isomorphic if and only if the groups G and H are isomorphic.

Theorem 24.

Let G be a group, $Z(G)$ its center i $m \geqslant 3$. Let also k and l be such that $1 \leqslant k < l \leqslant m+1$, $l-k > 1$. Then

1. *If $x \notin Z(G)$ then for every $y \in G$, such that $xy \neq yx$, the vertex $\mathbf{x}_{[k,l)}$ is the unique vertex $V(\mathbf{e}) \cap V(\mathbf{g})$, where*

$$\mathbf{g} = \mathbf{y}_{[i,k)}(\mathbf{yx})_{[k,j)}\mathbf{x}_{[j,l)}, \ (1 \leqslant i < k) \quad \text{or}$$

$$\mathbf{g} = \mathbf{x}_{[k,i)}(\mathbf{yx})_{[i,l)}\mathbf{y}_{[l,j)}, \ (l < j \leqslant m+1).$$

2. *If for some $\mathbf{a} \in \mathscr{G}_m(G)$, either $\mathbf{x}_{[k,k+1)} \in V(\mathbf{e}) \cap V(\mathbf{a})$" or $\mathbf{x}_{[1,m+1)} \in V(\mathbf{e}) \cap V(\mathbf{a})$, then $|V(\mathbf{e}) \cap V(\mathbf{a})| > 1$.*

Let G be a group, $Z(G)$ its center i $m \geqslant 3$. Let also k and l be such that $1 \leqslant k < l \leqslant m+1$, $l-k > 1$. Let A be the adjacency matrix of $\mathscr{G}_m(G)$. Then the $(\mathbf{g}, \mathbf{h})$-entry of A^2 is equal to the number of paths of length 2 from $\mathbf{g}$ to $\mathbf{h}$. Hence in the row (column) of A^2 labeled by $\mathbf{x}_{[k,l)}$ there are entries equal to 1 if and only if $x \notin Z(G)$,.

Theorem 25.

There exists a subgraph $\mathscr{Z}$ of the graph $\mathscr{G}_m(G)$ with the set of vertices equal to G^m, determined by the combinatorial structure of $\mathscr{G}_m(G)$ such that

1. *All connected components of $\mathscr{Z}$ are isomorphic to $\mathscr{G}_m(Z(G))$.*
2. *One can define adjacency relation on the set of connected components giving the graph isomorphic to $\mathscr{G}_m(G/Z(G))$.*

Definition

Let $\mathscr{D}_m(G)$ be a subgraph of $\mathscr{G}_m(G)$ with the set of vertices G^m. Two vertices are adjacent ($\mathbf{g} \sim \mathbf{h}$) in $\mathscr{D}_m(G)$, if there exists $\mathbf{x} \in S'$, such that $\mathbf{sg} = \mathbf{h}$, where

$$S' = \bigcup_{k=1}^{m-1} G_{[k,k+2)}.$$

Proposition 26.

For $m \geqslant 3$ the number of connected components of $\mathscr{D}_m(G)$ is equal $|G/G'|$. In particular $\mathscr{D}_m(G)$ is connected if and only if $G' = G$.

Proof.

$$(x^{-1}y^{-1}xy, e, e, e, \dots, e) =$$
$$= (x^{-1}y^{-1}, x^{-1}y^{-1}, e, e, \dots, e) \cdot (e, y, y, e, \dots, e)\cdot$$
$$(xy, xy, e, e, \dots, e) \cdot (e, y^{-1}, y^{-1}, e, \dots, e),$$

$$(e, x^{-1}y^{-1}xy, e, e, \dots, e) =$$
$$= (x^{-1}y^{-1}, x^{-1}y^{-1}, e, e, \dots, e) \cdot (e, x, x, e, \dots, e)\cdot$$
$$(yx, yx, e, e, \dots, e) \cdot (e, x^{-1}, x^{-1}, e, \dots, e),$$

$$(e, e, x^{-1}y^{-1}xy, e, \dots, e) =$$
$$= (x, x, e, e, \dots, e) \cdot (e, x^{-1}y^{-1}, x^{-1}y^{-1}, e, \dots, e)\cdot$$
$$(x^{-1}, x^{-1}, e, e, \dots, e) \cdot (e, xy, xy, e, \dots, e).$$

Proof c.d.

Then
$$(G')^m \leqslant \langle S' \rangle$$
On the other hand, if $G' < G$ then the functio $\varphi : G^n \to G/G'$ defined by the formula
$$\varphi(x_1, \ldots, x_m) = x_1 x_2^{-1} x_3 x_4^{-1} \cdots x_m^{(-1)^{m-1}} G'.$$
is a homomorphism of groups and
$$\langle S' \rangle \leqslant \mathrm{Ker}(\varphi).$$
Moreover, if $\varphi(x_1, \ldots, x_m) = e_{G/G'}$ Therefore $x_1 x_2^{-1} x_3 x_4^{-1} \cdots x_m^{(-1)^{m-1}} \in G'$. But
$$(x_1, x_2, \ldots, x_m) =$$
$$= (\mathbf{x}_1)_{[1,3)} (\mathbf{x}_1^{-1} \mathbf{x}_2)_{[2,4)} (\mathbf{x}_2^{-1} \mathbf{x}_1 \mathbf{x}_3)_{[3,5)} (\mathbf{x}_3^{-1} \mathbf{x}_1^{-1} \mathbf{x}_2 \mathbf{x}_4)_{[4,6)} \ldots,$$