Rings whose proper subrings are Lie nilpotent

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Let $A = (A, +, [-,-])$ be a Lie ring. A left-normed product

$$[x_1, \ldots, x_n, x_{n+1}],$$

where n is a positive integer and $x_1, \ldots, x_n, x_{n+1} \in A$, is defined recursively

$$[x_1, \ldots, x_n, x_{n+1}] := [[x_1, \ldots, x_n], x_{n+1}].$$

A Lie ring A is said to be **nilpotent** if there is a positive integer n such that

$$[x_1, x_2, \ldots, x_n] = 0$$

is true for any sequence $\{x_i\}_{i=1}^{\infty}$ in A.

Earlier, the possible structures of finite-dimensional Lie algebras A over a field $\mathbb{F}$ all of whose proper subalgebras are nilpotent have been studied by E.L. Stitzinger (1971), A.G. Gein, S.V. Kuznecov and Ju.N. Mukhin (1972), D. Towers (1980), A.G. Gein (1984, 1985, 1989), A. Elduque (1986) and others.
In this way A.A. Lashkhi and I. Zimmermann (2006), P. Zusmanovich (2014) and others have investigated Lie algebras with many nilpotent subalgebras. In particular, D. Towers has proved:

- if $\mathbb{F}$ is algebraically closed and A is minimal non-nilpotent, then A must be two-dimensional non-abelian,
- there are non-nilpotent soluble finite-dimensional Lie algebras of arbitrary dimension over the rational numbers field, and over any finite field, having all proper subalgebras nilpotent.

Thus a finitely generated minimal non-nilpotent Lie ring (i.e., a non-nilpotent Lie ring such that all of its proper subrings are nilpotent) exists. If a Lie ring A is not finitely generated and its any proper subring is nilpotent, then A is locally nilpotent (i.e., every finitely generated subring of A is nilpotent). In the case of Lie rings we obtain the following

Lemma.

If A is a minimal non-nilpotent Lie ring, then A^+ is a p-group.

Lemma.

A minimal non-nilpotent Lie ring A is countable.

$$* * *$$

By $[A, A]$ we denote the ideal of a Lie ring A generated by all commutators $[a, b]$, where $a, b \in A$. If $A \neq [A, A]$, then A is called **non-perfect**.

Let $Z_1(A) := Z(A)$ and

- $Z_{\alpha+1}(A)/Z_\alpha(A) \quad = \quad Z_1(A/Z_\alpha(A))$ if α is ordinal,
- $Z_\lambda(Z) \quad = \quad \cup_{\alpha<\lambda} Z_\alpha(A)$ if λ is limit ordinal.

If $A = Z_\beta(A)$ for some ordinal β, then we say that A is **hypercentral**.

Lemma.
Let A be a non-perfect minimal non-nilpotent Lie ring. Then A is not finitely generated if and only if A is hypercentral.

Lemma.
Let A be a non-perfect minimal non-nilpotent Lie ring. Then the following statements hold:

(i) if $Z(A) = 0$, then A is finite and A' is a minimal ideal of A,

(ii) if A is finitely generated, then A' is abelian or $Z(A)$ is nonzero.

We prove

Theorem.
A non-perfect Lie ring A is minimal non-nilpotent if and only if it is of one of the following types:

(*i*) $A = A' \propto \langle x \rangle_{lrg}$, $pA' = 0$, $\langle x \rangle_{lrg}^{+} \cong \mathbb{Z}_{p^m}$ ($m \geq 1$ is an integer), $A' = \sum_{n=1}^{\infty}{}^{\oplus} \langle a_n \rangle_{lrg}$ is abelian, $[a_1, x] = 0$, $[a_{n+1}, x] = a_n$ ($n \in \mathbb{N}$),

(*ii*) $A = A' \propto \langle x \rangle_{lrg}$, $pA' = 0$, $\langle x \rangle_{lrg}^{+} \cong \mathbb{Z}_{p^m}$ ($m \geq 1$ is an integer), $A' = A'' \oplus \langle e_1, \ldots, e_n \rangle_{gr}$ is an additive group direct sum, ad_x acts nilpotent on A'' and either A' is a minimal ideal of A or $A'' = Z(A)$.

In the sequel R is an associative ring (not necessary with unity). Every associative ring $R = (R, +, \cdot)$ can be viewed as a Lie ring $R^L = (R, +, [-,-])$ via the Lie multiplication $[a, b] = a \cdot b - b \cdot a$. By $[U, V]$ we will denote the additive subgroup of R generated by all $[u, v]$, where $u \in U$ and $v \in V$.
If the Lie ring R^L is nilpotent, then R is called **Lie nilpotent**.

An associative ring is called **minimal non-(Lie nilpotent)** if it is not Lie nilpotent but all its proper subrings are Lie nilpotent.
A group is called **minimal non-nilpotent** if it is not nilpotent but all its proper subgroups are nilpotent.

Lie nilpotent group algebras have been studied by I.B.S. Passi, D.S. Passman and S.K. Sehgal (1973) and Lie T-nilpotent group rings by A.A. Bovdi and I.I. Khripta (1986).
We establish that

Proposition.

If all proper subrings of a group ring $\mathbb{F}[G]$ of a group G over a field $\mathbb{F}$ are Lie nilpotent, then $\mathbb{F}[G]$ is Lie nilpotent.

Lemma.

Every minimal non-(Lie nilpotent) ring R with unity is local of power prime characteristic and $R/J(R)$ is finite.

Recall that a unitary ring R is called **local** if $R/J(R)$ is a skew field.

Examples.
Let $\mathbb{F}_4 = \{0, 1, a, b\}$ be a field consists of 4 elements such that $ab = 1 = ba,\ a^2 = b,\ a + b = 1$ and $\sigma : \mathbb{F}_4 \ni w \mapsto w^2 \in \mathbb{F}_4$ be a non-trivial field automorphism.
a) We will consider the quotient ring of the skew polynomial ring $\mathbb{F}_4[X; \sigma]$ (with a ring multiplication induced by the rule $Xw = w^{\sigma} X$) by the principal ideal $\langle X^2 \rangle$

$$R := \mathbb{F}_4[X; \sigma]/\langle X^2 \rangle = \mathbb{F}_4 + \mathbb{F}_4 u.$$

Then $u^2 = 0$, R has 7 proper subrings:

$$0,\ \mathbb{F}_2,\ \mathbb{F}_4,\ \mathbb{F}_2 u,\ \mathbb{F}_4 u,\ \mathbb{F}_2 + \mathbb{F}_2 u,\ \mathbb{F}_2 + \mathbb{F}_4 u$$

and all those proper subrings are Lie nilpotent.

First we find that

$$[u,a] = ua - au = (a^\sigma + a)u = u = [u,b]$$

and $[[R,R],R] \ni [u,a] = u$ and then we conclude that

$$\gamma_3(R) = [[R,R],R] = [R,R] = \gamma_2(R).$$

Thus R is not Lie nilpotent. This means that R is minimal non-(Lie nilpotent). Moreover, R^L is a minimal non-nilpotent Lie ring and the unit group $U(R)$ is a minimal non-nilpotent group of order 12.
b) If

$$D := \mathbb{F}_4[X;\sigma]/\langle X^4\rangle = \mathbb{F}_4 + \mathbb{F}_4 t + \mathbb{F}_4 t^2 + \mathbb{F}_4 t^3,$$

where $t^4 = 0$, then $[t^3, a] = (\sigma^3(a) - a)t^3 = t^3$ and, consequently, the subring $\mathbb{F}_4 + \mathbb{F}_4 t^3$ is not Lie nilpotent in D.

c) Assume that

$$B := \mathbb{F}_4[X;\sigma]/\langle X^3\rangle = \mathbb{F}_4 + \mathbb{F}_4 v + \mathbb{F}_4 v^2,$$

where $v^3 = 0$. Then $[v, a] = v = [v, b]$, $[av, bv] = v^2$ and so the commutator ideal $C(B) = \mathbb{F}_4 v + \mathbb{F}_4 v^2$ is non-commutative and of the nilpotency index 3. Moreover, $\gamma_2(B) = \gamma_3(B)$ and B is minimal non-(Lie nilpotent).

Recall that I.S. Cohen (1949) first found that any commutative complete (on the $J(R)$-adic topology) Noetherian local ring R contains a coefficient subring. W.E. Clark (1972) confirmed this it in the case of (non-commutative) finite local ring, and T. Sumiyama (1995) in the case when R is a local ring of characteristic p^n with the nilpotent Jacobson radical $J(R)$ and the residue field $R/J(R)$ algebraic over the field $\mathbb{F}_p$.

$* * *$

Recall that a local ring R of characteristic p^n has **a coefficient subring** S if S is a commutative subring of R, $R = J(R) + S$, $J(S) = S \cap J(R) = pS$ and $R/J(R) \cong S/pS$.

We prove

Theorem.

Let R be a finite ring. Then R is a minimal non-(Lie nilpotent) ring if and only if it is a finite local ring of characteristic p^n which contains a coefficient subring S, its unit group $U(R) = (1 + J(R)) \rtimes \langle b \rangle$ is a group semidirect product of a normal p-subgroup $1 + J(R)$ and a cyclic subgroup $\langle b \rangle \cong U(R/J(R))$ of order $p^{q^m} - 1$ (that is subfields of $R/J(R)$ are linearly ordered), $M := J(R)^2 + pS = C_{J(R)}(b)$ is an ideal of R such that $J(R)/M$ is a minimal ideal of R/M, $|J(R)/M| = p^{q^m}$ and $b^s \in Z(U(R))$, where $s = \frac{p^{q^m}-1}{p^{q^{m-1}}-1}$, p, q are primes, $n, m \geq 1$ are integers and one of the following holds:

(*i*) $J(R)^2 = 0$,

(*ii*) $J(R)$ is of the nilpotency index 3 and $q = 2$.

In particular case we have the following

Corollary.

Let R be a finite ring of prime characteristic p and $J(R)^2 = 0$. Then the following conditions are equivalent:

(1) R is a minimal non-(Lie nilpotent) ring,

(2) $R = J(R) \oplus S$ is a group direct sum, where $S \cong \mathbb{F}_{q^{p^n}}$, q and q are primes, $n, m \geq 1$ are integers and $J(R)$ is a minimal ideal of R,

(3) $R \cong \mathbb{F}_{q^{p^n}}[X; \sigma]/\langle X^2 \rangle$, where p and q are primes, $n, m \geq 1$ are integers and $\sigma : \mathbb{F}_{q^{p^n}} \to \mathbb{F}_{q^{p^n}}$ is a non-trivial automorphism of $\mathbb{F}_{q^{p^n}}$.

Remark.

Notice that if R is a minimal non-(Lie nilpotent) ring with the torsion unit subgroup $U(R)$ which satisfies one of the following conditions:

(a) R is right Artinian,

(b) R is right Goldie,

(c) R satisfies the ascending chain condition on both left and right annihilators,

then R is finite.

An associative ring R is a monoid with respect to a circle multiplication "$\circ$" defined by the rule $a \circ b = a + b + a \cdot b$ for all $a, b \in R$. The set $R^\circ = \{a \in R \mid \text{ there exists } b \in B \text{ such that } a \circ b = 0 = b \circ a\}$ is a group (so-called *the adjoint group* of R). If $R^\circ = R$, then R is called **a (Jacobson) radical ring**. We also study radical rings with Lie nilpotent proper subrings and prove the following theorem.

Theorem.
Let R be a radical ring and its every proper subring be Lie nilpotent. Then the following conditions are true:

(*i*) if R is 2-torsion-free and $[R, R]$ is proper in R, then R is Lie nilpotent,

(*ii*) if $R = [R, R]$, then all its one-sided ideals are nilpotent and R^L is a minimal non-nilpotent Lie ring.

In the case when all subgroups are nilpotent in the adjoint group of a radical ring we obtain the following

Corollary.

Let R be a 2-torsion-free radical ring. If all proper subgroups of the adjoint group R° are nilpotent, then either R is Lie nilpotent or R° is perfect (i.e., its derived subgroup is not proper); if, moreover, R° is torsion (respectively locally graded), then R is a Lie nilpotent nil ring.

I.L. Hmel'nickiĭ (1971) has proved that *an associative ring with nilpotent proper subrings is either nilpotent or it is isomorphic to* $\mathbb{Z}_{p^k}$.

A field F satisfies the Brauer condition if there exists a function $\psi(d)$ with positive integer values such that, for every integer $d > 0$ and any nonzero elements $a_1, \ldots, a_{\psi(d)}$, there are $x_1, \ldots, x_{\psi(d)}$ (some of which are nonzero) satisfying

$$a_1 x_1^d + a_2 x_2^d + \cdots + a_{\psi(d)} x_{\psi(d)}^d = 0.$$

$$* * *$$

Every algebraically field and every finite field satisfies the Brauer condition. The rational numbers field not satisfies the Brauer condition.

$$* * *$$

I.V. L'vov (1971) has proved: *Let F be a field satisfying the Brauer condition. If R is an associative F-algebra with every proper subalgebra to be nilpotent, then either R is nilpotent or it is an one-dimensional F-algebra.*

As it is well known, there exist (commutative) non-nilpotent algebras with nilpotent proper subalgebras). But it is true the following

Theorem.

Let R be an associative algebra over a field $\mathbb{F}$ of characteristic 0 with nilpotent proper subalgebras. Then R contains a nilpotent ideal I such that R/I is commutative (and so R is Lie soluble). If, moreover, R is not finitely generated, then it is Lie nilpotent.

Corollary.

An associative algebra over a field $\mathbb{F}$ of characteristic 0 with nilpotent proper subalgebras is nilpotent if and only if is nilpotent every commutative $\mathbb{F}$-algebra with nilpotent proper subalgebras.

Thank you!