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REGULAR CLASSES OF LINEAR EXTENSIONS OF DYNAMICAL SYSTEMS ON A TORUS

Abstract. The paper presents new methods of construction of the Lyapunov function for some sets of linear extensions of dynamic systems on torus.The paper is divided into two parts. The first part contains a brief theoretical introduction. In the second part are presented results of study and examples.

1. Introduction

Consider the system of differential equations

$$\frac{d\phi}{dt} = a(\phi), \qquad \frac{dx}{dt} = A(\phi)x, \tag{1}$$

where $\phi \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $a(\phi)$ is a continuous, 2π-periodic function with respect to each variable, ϕ_j, $j = \overline{1,m}$, vector function. $A(\phi)$ is a $n \times n$-dimensional matrix, whose elements are continuous, 2π-periodic with respect to each variable vector functions. For a function $a(\phi)$ we additionally assume, that it satisfies the

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Lipschitz condition, which implies that the Cauchy problem

$$\frac{d\phi}{dt} = a(\phi), \qquad \phi \mid_{t=0} = \phi_0,$$

has exactly one solution $\phi_t(\phi_0)$ for every value ϕ_0.

In this article we will use the following notation: $C^0(T_m)$ – the space of real functions, continuous and bounded on the m-dimensional torus; $C^1(T_m)$ – the subspace of $C^0(T_m)$, that is all continuously differentiable functions on the m-dimensional torus; $\langle x, y\rangle = \sum\limits_{i=1}^{n} x_i y_i$ – the inner product in $\mathbb{R}^n$; $\|x\| = \sqrt{\langle x, x\rangle}$ – the norm of $x \in \mathbb{R}^n$; $\Omega_\tau^t(\phi)$ – a fundamental matrix of a linear system of equations $\frac{dx}{dt} = A(\phi)x$, normed at $t = \tau$, i.e. $\Omega_\tau^t(\phi) \mid_{t=\tau} = I_n$, where I_n is a $n \times n$-dimensional unit matrix. The matrix $\Omega_\tau^t(\phi)$ has the following property $[\Omega_\tau^t(\phi)]^{-1} = \Omega_t^\tau(\phi)$ [2,4].

One of the main and most important problems in the study of the systems (1) is the problem of finding a Green function. Let us to recall a definitions of Green function and regularity of system (1) [1, 2].

Definition 1. *System (1) has the Green function $G_0(\tau, \phi)$ of the problem of the bounded invariant manifold if there exists a continuous $n \times n$-dimensional matrix $C(\phi)$, such that the function*

$$G_0(\tau, \phi) = \begin{cases} \Omega_\tau^0(\phi) \cdot C(\phi_\tau(\phi)), & \tau \leq 0, \\ \Omega_\tau^0(\phi) \cdot [C(\phi_\tau(\phi)) - I_n], & \tau > 0, \end{cases}$$

satisfies the estimation

$$\|G_0(\tau, \phi)\| \leq K \cdot exp\{-\gamma|\tau|\},$$

where K and γ are positive constants. The function $G_0(\tau, \phi)$ is said to be the Green function of the problem of the bounded invariant manifold of the system (1).

Definition 2. *System (1) is called regular, if and only if it has exactly one Green function to the problem of bounded invariant manifold of the system (1).*

Effective method to determine the existence of the Green function for the system (1) is the method of generalized Lyapunov function. It is known [1], that if there exists a quadratic form

$$V = \langle S(\phi), x\rangle, \qquad S(\phi) \equiv S^T(\phi) \in C^1(T_m), \tag{2}$$

which the derivative related to the system (1) is positive definite, i.e.:

$$\dot{V} = \left\langle \left[\sum_{j=1}^{m} \frac{\partial S(\phi)}{\partial \phi_j} a_j(\phi) + S(\phi)A(\phi) + A^T(\phi)S(\phi) \right] x, x \right\rangle \geq \gamma \|x\|^2, \qquad (3)$$

$\gamma = \text{const} > 0$, with assumption, that $\det S(\phi) \neq 0$, $\forall \phi \in T_m$, then the system (1) is regular. The quadratic form (2) is often called the generalized Lyapunov function.

Together with the system (1) we often consider the conjugated system

$$\frac{d\phi}{dt} = a(\phi), \qquad \frac{dy}{dt} = -A^T(\phi)y. \qquad (4)$$

When for the conjugated system (4) exists a quadratic form $W = \langle \overline{S}(\phi)y, y \rangle$, which the derivative related to the system (4) is positive definite, and $\det \overline{S}(\phi) = 0$ for some values of $\phi \in \mathbb{R}^m$, then the system (1) has many different Green functions. When $\det \overline{S}(\phi) \neq 0$, then the conjugated system (4) is regular.

2. Main results

The research on finding the quadratic form (2), which satisfies the inequality (3) is always a very difficult task. Therefore, it is proposed to extract some classes of the system (1) for which we can find the right quadratic form (2). In this paper will be present some criteria for regularity of the system (1) by using generalized Lyapunov function. More information about generalized Lyapunov function can be found in [3]. This section also presents some interesting examples.

Consider the systems of differential equations

$$\frac{d\phi}{dt} = \sin \phi, \qquad \frac{dx}{dt} = (\cos 2n\phi)\, x, \qquad n = 1, 2, 3, \ldots \qquad (5)$$

For systems (5) the generalized Lyapunov function has the form

$$V = x^2 \exp\{s_n(\phi)\}, \qquad n = 1, 2, 3, \ldots,$$

where $s_1(\phi) = -4\cos\phi$, $s_2(\phi) = -\frac{16}{3}\cos^3\phi$, $s_3(\phi) = -\frac{64}{5}\cos^5\phi + \frac{32}{3}\cos^3\phi - 4\cos\phi$, etc.

The research on the systems

$$\frac{d\phi}{dt} = \sin\phi, \qquad \frac{dx}{dt} = (\cos(2n+1)\phi)\, x, \qquad n = 1, 2, 3, \ldots, \tag{6}$$

lead us to the following conclusions: every system (6) has many different Green functions, which means that for the conjugated systems

$$\frac{d\phi}{dt} = \sin\phi, \qquad \frac{dy}{dt} = -\,(\cos(2n+1)\phi)\, y, \qquad n = 1, 2, 3, \ldots,$$

we can choose the quadratic form

$$V = y^2 \exp\{s_n(\phi)\}\,(-\cos\phi)\,, \qquad n = 1, 2, 3, \ldots,$$

where $s_1(\phi) = 4\cos^2\phi$, $s_2(\phi) = 5\cos^4\phi + \frac{5}{2}\sin^4\phi$, etc.

The generalization of the system (5) lead us to the system

$$\frac{d\phi}{dt} = a(\phi), \qquad \frac{dx}{dt} = [\mu_0(\phi) + \mu_1(\phi)]x, \tag{7}$$

where $\phi = (\phi_1, \phi_2, \ldots, \phi_m)$, $x \in \mathbb{R}$, $\mu_j(\phi) = \mu_j(\phi_1, \phi_2, \ldots, \phi_m)$ is a continuous scalar function, $\mu_j(\phi) \in C^0(T_m)$, $j = 0, 1$. We assume that function $\mu_1(\phi)$ is such that the following partial differential equation

$$\sum_{j=1}^{m} \frac{\partial s}{\partial \phi_j} a_j(\phi) = \mu_1(\phi_1, \phi_2, \ldots, \phi_m)\,, \tag{8}$$

has a solution $s = s(\phi) = s(\phi_1, \phi_2, \ldots, \phi_m)$, which is 2π-periodic function with respect to each variable ϕ_j, $j = \overline{1, m}$. By calculating the derivative of the function

$$V = x^2 \exp\{-2s(\phi)\}\,, \tag{9}$$

related to the system (7) we receive

$$\begin{aligned}\dot{V} = 2x\dot{x}\exp\{-2s(\phi)\} - x^2 2\dot{s}(\phi)\exp\{-2s(\phi)\} = \\ = 2x^2\left[\mu_0(\phi) + \mu_1(\phi) - \dot{s}(\phi)\right]\exp\{-2s(\phi)\} = \\ = 2x^2\left[\mu_0(\phi)\right]\exp\{-2s(\phi)\}\,,\end{aligned} \tag{10}$$

where $\dot{s} = \sum\limits_{j=1}^{m} \frac{\partial s}{\partial \phi_j} a_j(\phi)$. It brings us to the following statement.

Lemma 3. *If a scalar function $\mu_0(\phi) \in C^0(T_m)$ in the system (7) satisfies the inequality*

$$\mu_0(\phi) > 0 \qquad \forall \phi \in T_m, \tag{11}$$

and the equation (8) has a solution $s(\phi) \in C^1(T_m)$, then the derivative of function (9) related to the system (7) will be positive definite which means that the system (7) is regular.

Remark 4. In Lemma 3, the inequality (11) can be replaced by

$$|\mu_0(\phi)| > 0 \qquad \forall \phi \in T_m, \tag{12}$$

and when $\mu_0(\phi) < 0 \ \forall \phi \in T_m$, we can multiply function (9) by -1 and then its derivative will be positive definite.

Very useful is the following remark.

Remark 5. If it is impossible to find a solution $s(\phi) = s(\phi_1, \phi_2, \ldots, \phi_m)$ of the equation (8) or such solution does not exist then we can consider the equation of the form

$$\sum_{j=1}^{m} \frac{\partial s}{\partial \phi_j} a_j(\phi) = \mu_1(\phi) - \overline{\mu}(\phi), \tag{13}$$

where we choose the function $\overline{\mu} \in C^1(T_m)$ such that the equation (13) has a solution $s(\phi) \in C^1(T_m)$, then when the right hand side of equation (13) satisfies the inequality

$$|\mu_0(\phi) - \overline{\mu}(\phi)| > 0 \qquad \forall \phi \in T_m,$$

the system (7) is regular.

Now let analyse some interesting examples.

Example 6. Consider the system of differential equations

$$\begin{cases} \frac{d\phi}{dt} = (1 + \varepsilon_1 \cos\phi + \varepsilon_2 \sin\phi)^{-1}, \\ \frac{dx}{dt} = \Big(\mu_0(\phi) + \sum\limits_{k=1}^{n} a_k \cos k\phi + b_k \sin k\phi\Big) x, \end{cases} \tag{14}$$

where $\varepsilon_1, \varepsilon_2, a_k, b_k$ are constants for every natural k, $\varepsilon_1^2 + \varepsilon_2^2 < 1$. Now we can ask a question: what conditions has to satisfy the function $\mu_0(\phi) \in C^0(T_1)$ in order to the regularity of the system (14)?

Let $\mu_1(\phi)$ and $\overline{\mu}(\phi)$ has the form

$$\mu_1(\phi) = \sum_{k=1}^{n} a_k \cos k\phi + b_k \sin k\phi, \tag{15}$$

$$\overline{\mu}(\phi) = \frac{a_1\varepsilon_1 + b_1\varepsilon_2}{2\left(1 + \varepsilon_1 \cos\phi + \varepsilon_2 \sin\phi\right)}. \tag{16}$$

Then the second equation in system (14) can be written as

$$\frac{dx}{dt} = \left[(\mu_0(\phi) + \overline{\mu}(\phi)) + (\mu_1(\phi) - \overline{\mu}(\phi))\right] x.$$

Now we check that the equation $\dot{s} = \mu_1(\phi) - \overline{\mu}(\phi)$, has a solution $s = s(\phi) \in C^1(T_1)$. By considering the equality (15) we have

$$\begin{aligned}\dot{s} = \frac{ds}{d\phi} \cdot \frac{d\phi}{dt} = \frac{ds}{d\phi} \cdot \frac{1}{1 + \varepsilon_1 \cos\phi + \varepsilon_2 \sin\phi} = \mu_1(\phi) - \overline{\mu}(\phi) = \\ = \sum_{k=1}^{n} \left(a_k \cos k\phi + b_k \sin k\phi\right) - \frac{a_1\varepsilon_1 + b_1\varepsilon_2}{2\left(1 + \varepsilon_1 \cos\phi + \varepsilon_2 \sin\phi\right)},\end{aligned} \tag{17}$$

from where we get

$$\begin{aligned}\frac{ds}{d\phi} = \left(1 + \varepsilon_1 \cos\phi + \varepsilon_2 \sin\phi\right) \mu_1(\phi) - \frac{1}{2}\left(a_1\varepsilon_1 + b_1\varepsilon_2\right) = \\ = \mu_1(\phi) + \left(\varepsilon_1 \cos\phi + \varepsilon_2 \sin\phi\right) \cdot \\ \cdot \left(a_1\varepsilon_1 + b_1\varepsilon_2\right) + \left(\varepsilon_1 \cos\phi + \varepsilon_2 \sin\phi\right) \cdot \sum_{k=2}^{n} \left(a_k \cos k\phi + b_k \sin k\phi\right) - \frac{1}{2}\left(a_1\varepsilon_1 + b_1\varepsilon_2\right) = \\ = \mu_1(\phi) + \frac{1}{2}\left(a_1\varepsilon_1 + b_1\varepsilon_2\right) \cos 2\phi + \frac{1}{2}\left(a_1\varepsilon_1 + b_1\varepsilon_2\right) \sin 2\phi + \\ + \left(\varepsilon_1 \cos\phi + \varepsilon_2 \sin\phi\right) \cdot \sum_{k=2}^{n} a_k \left(\cos k\phi + b_k \sin k\phi\right).\end{aligned}$$

It is obvious that the integral of this function is a periodic function $s = s_0(\phi)$ and function $\overline{\mu}(\phi)$ can be estimated as follows

$$\frac{a_1\varepsilon_1 + b_1\varepsilon_2}{2\left(1 + \sqrt{\varepsilon_1^2 + \varepsilon_2^2}\right)} \leq \overline{\mu}(\phi) \leq \frac{a_1\varepsilon_1 + b_1\varepsilon_2}{2\left(1 - \sqrt{\varepsilon_1^2 + \varepsilon_2^2}\right)}, \qquad \text{when} \quad a_1\varepsilon_1 + b_1\varepsilon_2 > 0,$$

$$\frac{a_1\varepsilon_1+b_1\varepsilon_2}{2\left(1-\sqrt{\varepsilon_1^2+\varepsilon_2^2}\right)} \le \overline{\mu}(\phi) \le \frac{a_1\varepsilon_1+b_1\varepsilon_2}{2\left(1+\sqrt{\varepsilon_1^2+\varepsilon_2^2}\right)}, \qquad \text{when} \quad a_1\varepsilon_1+b_1\varepsilon_2<0.$$

Considering the inequality $\mu_0(\phi)-\overline{\mu}(\phi)>0$, we get a sufficient condition of regularity of the system (14):

$$\mu_0(\phi) > \begin{cases} \frac{a_1\varepsilon_1+b_1\varepsilon_2}{2\left(1+\sqrt{\varepsilon_1^2+\varepsilon_2^2}\right)}, & a_1\varepsilon_1+b_1\varepsilon_2>0, \\ \frac{a_1\varepsilon_1+b_1\varepsilon_2}{2\left(1-\sqrt{\varepsilon_1^2+\varepsilon_2^2}\right)}, & a_1\varepsilon_1+b_1\varepsilon_2<0. \end{cases}$$

Then considering the opposite inequality $\mu_0(\phi)-\overline{\mu}(\phi)<0$, we get another sufficient condition of regularity of the system (14):

$$\mu_0(\phi) > \begin{cases} \frac{a_1\varepsilon_1+b_1\varepsilon_2}{2\left(1-\sqrt{\varepsilon_1^2+\varepsilon_2^2}\right)}, & a_1\varepsilon_1+b_1\varepsilon_2>0, \\ \frac{a_1\varepsilon_1+b_1\varepsilon_2}{2\left(1+\sqrt{\varepsilon_1^2+\varepsilon_2^2}\right)}, & a_1\varepsilon_1+b_1\varepsilon_2<0. \end{cases}$$

Example 7. Consider the system of differential equations

$$\begin{cases} \frac{d\phi_1}{dt} = 1+2\sin\phi_1+3\sin\phi_2, \\ \frac{d\phi_2}{dt} = 2+3\sin\phi_1+4\sin\phi_2, \\ \frac{dx}{dt} = [\mu_0(\phi_1,\phi_2)+5\sin\phi_1+6\sin\phi_2+ \\ \qquad\qquad +7\sin^2\phi_1+8\sin\phi_1\sin\phi_2+9\sin^2\phi_2]\,x. \end{cases} \tag{18}$$

The equation which is corresponding to (13) has the form

$$\begin{aligned} &\frac{\partial s}{\partial\phi_1}(1+2\sin\phi_1+3\sin\phi_2)+\frac{\partial s}{\partial\phi_2}(2+3\sin\phi_1+4\sin\phi_2)= \\ &\quad =5\sin\phi_1+6\sin\phi_2+7\sin^2\phi_1+8\sin\phi_1\sin\phi_2+9\sin^2\phi_2-\overline{\mu}(\phi_1,\phi_2). \end{aligned} \tag{19}$$

Now we want to find a function of the form

$$\overline{\mu}(\phi_1,\phi_2)=\overline{\mu}_1\sin\phi_1+\overline{\mu}_2\sin\phi_2+\overline{\mu}_{11}\sin^2\phi_1+\overline{\mu}_{12}\sin\phi_1\sin\phi_2+\overline{\mu}_{22}\sin^2\phi_2, \tag{20}$$

where $\overline{\mu}_1,\overline{\mu}_2,\overline{\mu}_{11},\overline{\mu}_{12},\overline{\mu}_{22}$ are constants such that the equation (19) has a solution of the form

$$s=s(\phi_1,\phi_2)=s_1\cos\phi_1+s_2\cos\phi_2, \qquad s_1,s_2=\text{const}. \tag{21}$$

By substituting the equations (20) and (21) to the equation (19) we get the following equalities

$$-s_1 = 5 - \overline{\mu}_1, \qquad -2s_2 = 6 - \overline{\mu}_2, \qquad -2s_1 = 7 - \overline{\mu}_{11},$$
$$-3(s_1 + s_2) = 8 - \overline{\mu}_{12}, \qquad -4s_1 = 9 - \overline{\mu}_{22}.$$

Based on these equalities we obtain the trigonometric polynomial with parameters s_1, s_2:

$$\overline{\mu}(\phi_1, \phi_2) = (5 + s_1)\sin\phi_1 + (6 + s_2)\sin\phi_2 + (7 + s_1)\sin^2\phi_1 + \\ + (8 + 3s_1 + 3s_2)\sin\phi_1 \sin\phi_2 + (9 + s_2)\sin^2\phi_2. \quad (22)$$

Thus, by substituting the polynomial (22) to the right side of the equality (19) the obtained equation will have the solution (21). By denoting $\sin\phi_i = \sigma_i$ the right hand side of (22) can be written as

$$\Phi(\phi_1, \phi_2, \sigma_1, \sigma_2) = (5 + s_1)\sigma_1 + (6 + s_1)\sigma_2 + (7 + s_1)\sigma_1^2 + \\ + (8 + 3s_1 + 3s_2)\sigma_1\sigma_2 + (9 + s_2)\sigma_2^2. \quad (23)$$

It is obvious that for arbitrary values of $s_1, s_2 \in \mathbb{R}$ there exists the maximal and the minimal value of the function (23):

$$\begin{aligned} \max_{|\sigma_i| \le 1} \Phi(\phi_1, \phi_2, \sigma_1, \sigma_2) &= \Phi_+(s_1, s_2), \\ \min_{|\sigma_i| \le 1} \Phi(\phi_1, \phi_2, \sigma_1, \sigma_2) &= \Phi_-(s_1, s_2). \end{aligned} \quad (24)$$

For example, if we choose $s_1 = 1, s_2 = 0$, then $\Phi_+(1, 0) = 41, \Phi_-(1, 0) = -\frac{254}{167}$. It means that the function (23) with parameters $s_1 = 1, s_2 = 0$ satisfies the inequalities $\overline{\mu}(\phi_1, \phi_2) \ge -\frac{254}{167}$, $\overline{\mu}(\phi_1, \phi_2) \le 41$. From this follows the sufficient condition of regularity of the system (18). For example, function $\mu_0(\phi_1, \phi_2)$ satisfies the inequality $\mu_0(\phi_1, \phi_2) > \frac{254}{167}$.

Remark 8. The equalities (24) are useful in selection of function (22): $\Phi_-(s_1, s_2) \le \overline{\mu}(\phi_1, \phi_2) \le \Phi_+(s_1, s_2)$. From this implies the sufficient conditions of regularity of the system (14): $\mu_0(\phi_1, \phi_2) > -\Phi_-(s_1, s_2)$ or $\mu_0(\phi_1, \phi_2) < \Phi_+(s_1, s_2)$, for some constant $s_1, s_2 \in \mathbb{R}$.

Some generalization of system (7) can be written as

$$\frac{d\phi}{dt} = a(\phi), \qquad \frac{dx}{dt} = [\mu_0(\phi) + \mu_1(\phi)]\, A(\phi)x, \tag{25}$$

where $x \in \mathbb{R}^n$, $A(\phi)$ is $n \times n$-dimensional matrix, $A(\phi) \in C^0(T_m)$.

Lemma 9. *Let the matrix $A(\phi)$ in the system (25) can be written in the form*

$$A(\phi) = \lambda I_n + \tilde{A}(\phi), \tag{26}$$

where $\lambda = const \neq 0$, $\tilde{A}^T(\phi) \equiv -\tilde{A}(\phi)$ and the equation (8) has solution of the form $s = s_0(\phi) \in C^1(T_m)$, that the derivative of the function

$$V = \|x\|^2 \exp\{-2\lambda s_0(\phi)\}, \tag{27}$$

related to the system (25) satisfies the inequality

$$\dot{V} = 2\lambda\mu_0(\phi)\|x\|^2 \exp\{-2\lambda s_0(\phi)\} > 0, \tag{28}$$

then system (25) is regular.

Proof. The quadratic form (27) can be written as

$$V = \langle x, x\rangle \exp\{-2\lambda s_0(\phi)\}.$$

By calculating the derivative of function (27) related to the system (25) and taking into account the form of the matrix $A(\phi)$ $(A(\phi) + A^T(\phi) = 2\lambda)$ we receive

$$\begin{aligned}
\dot{V} &= [\langle \dot{x}, x\rangle + \langle x, \dot{x}\rangle] \exp\{-2\lambda s_0(\phi)\} + \langle x, x\rangle \{-2\lambda \dot{s}_0(\phi)\} \exp\{-2\lambda s_0(\phi)\} = \\
&= \exp\{-2\lambda s_0(\phi)\} (\langle [\mu_0(\phi) + \mu_1(\phi)]\, A(\phi)x, x\rangle + \langle x, [\mu_0(\phi) + \mu_1(\phi)]\, A(\phi)x\rangle + \\
&\qquad + \langle x, x\rangle \{-2\lambda \dot{s}_0(\phi)\}) = \\
&= \exp\{-2\lambda s_0(\phi)\} ([\mu_0(\phi) + \mu_1(\phi)] (\langle A(\phi)x, x\rangle + \langle x, A(\phi)x\rangle) + \\
&\qquad + \langle x, x\rangle \{-2\lambda \dot{s}_0(\phi)\}) = \\
&= \exp\{-2\lambda s_0(\phi)\} \big([\mu_0(\phi) + \mu_1(\phi)] \langle (A(\phi) + A^T(\phi))x, x\rangle + \\
&\qquad + \langle x, x\rangle \{-2\lambda \dot{s}_0(\phi)\}) = \\
&= \exp\{-2\lambda s_0(\phi)\} (2\lambda [\mu_0(\phi) + \mu_1(\phi)] (\langle x, x\rangle + \langle x, x\rangle) + \\
&\qquad + \langle x, x\rangle \{-2\lambda \dot{s}_0(\phi)\}) =
\end{aligned}$$

$$= \exp\{-2\lambda s_0(\phi)\}\,\|x\|^2[2\lambda\mu_0(\phi) + 2\lambda\mu_1(\phi) - 2\lambda\dot{s}_0(\phi)] = \\ = 2\lambda\mu_0(\phi)\|x\|^2 \exp\{-2\lambda s_0(\phi)\}\,.$$

According to Lemma 3 and Remark 4 system (25) is regular. $\square$

Now consider the following system of differential equations

$$\begin{cases} \frac{d\phi}{dt} = a(\phi) \\ \frac{dx_1}{dt} = [\mu_{10}(\phi) + \mu_1(\phi)]\left[\lambda_1 I_{n_1} + \tilde{A}_1(\phi)\right] x_1, \\ \frac{dx_2}{dt} = [\mu_{20}(\phi) + \mu_2(\phi)]\left[\lambda_2 I_{n_2} + \tilde{A}_2(\phi)\right] x_2, \end{cases} \tag{29}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $\tilde{A}_i^T(\phi) \equiv -\tilde{A}_i(\phi)$, $\lambda_i \neq 0$, $i = 1, 2$.

Based on the Lemma 9 we get the following statement.

Theorem 10. *Assume that in the system (29) the scalar functions* $\mu_i(\phi) \in C^0(T_m)$, $i = 1, 2$, *are such that the following two partial differential equations*

$$\sum_{j=1}^{m} \frac{\partial s}{\partial \phi_j} a_j(\phi) = \mu_i(\phi_1, \phi_2, \ldots, \phi_m), \qquad i = 1, 2,$$

have the solutions $s = s_{0i}(\phi) \in C^1(T_m)$, $i = 1, 2$ *and also assume that* $\lambda_i = const \neq 0$, $\mu_{i0}(\phi) \neq 0$, $i = 1, 2$, *for every skew-symmetric matrices* $\tilde{A}_i(\phi) \in C^0(T_m)$, $\tilde{A}_i^T \equiv -\tilde{A}_i$, $i = 1, 2$. *Then the system (29) will be regular.*

The Theorem 10 comes from the fact that the derivative of quadratic form

$$V = \begin{cases} \|x_1\|^2 \exp\{-2\lambda_1 s_{01}(\phi)\} + & \mu_{10}(\phi) > 0 \wedge \lambda_2\mu_{20}(\phi) > 0, \\ \quad +\|x_2\|^2 \exp\{-2\lambda_2 s_{02}(\phi)\}, & \\ \|x_1\|^2 \exp\{-2\lambda_1 s_{01}(\phi)\} - & \mu_{10}(\phi) > 0 \wedge \lambda_2\mu_{20}(\phi) < 0, \\ \quad -\|x_2\|^2 \exp\{-2\lambda_2 s_{02}(\phi)\}, & \\ -\|x_1\|^2 \exp\{-2\lambda_1 s_{01}(\phi)\} + & \mu_{10}(\phi) < 0 \wedge \lambda_2\mu_{20}(\phi) > 0, \\ \quad +\|x_2\|^2 \exp\{-2\lambda_2 s_{02}(\phi)\}, & \\ -\|x_1\|^2 \exp\{-2\lambda_1 s_{01}(\phi)\} - & \mu_{10}(\phi) < 0 \wedge \lambda_2\mu_{20}(\phi) < 0. \\ \quad -\|x_2\|^2 \exp\{-2\lambda_2 s_{02}(\phi)\}, & \end{cases}$$

related to the system (29) is positive definite. The derivative of the above quadratic form is calculated analogically to the derivative of quadratic form in the proof of Lemma 9 (in each of 4 cases).

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