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ON REGULARITY OF LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

Abstract. The paper presents the theorems about regularity of systems of differential equations. The paper is divided into two parts. The first part contains a theoretical introduction. The second part contains theorems which allows to determine the regularity of the system using the generalized Lyapunov function.

1. Introduction

Consider the linear, homogeneous system of differential equations

$$\dot{x} = A(t)x, \tag{1}$$

where $\dot{x} = \frac{dx}{dt}$, $x \in \mathbb{R}^n$, $A(t)$ is a $n \times n$-dimensional matrix which entries are continuous and bounded on $\mathbb{R}$ functions.

Together with the system (1) we also consider the nonhomogeneous system of differential equations

$$\dot{x} = A(t)x + f(t), \tag{2}$$

where $f(t)$ is a continuous and bounded on $\mathbb{R}$ function what can be written as $f(t) \in C^0(\mathbb{R})$.

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In this paper we will use the following notation: $C^1(\mathbb{R})$ – the subspace of $C^0(\mathbb{R})$, that contains all continuously differentiable functions; $\langle x, y\rangle = \sum_{i=1}^{n} x_i \cdot y_i$ – the inner product in $\mathbb{R}^n$; $\|x\| = \sqrt{\langle x, x\rangle}$ – the norm in $\mathbb{R}^n$; Ω_τ^t – a fundamental matrix of a linear system $\frac{dx}{dt} = A(t)x$, normed at $t = \tau$, i.e. $\Omega_\tau^t|_{t=\tau} = I_n$, where I_n a is $n \times n$-dimensional unit matrix.

Definition 1. *The system (1) is called regular on* $\mathbb{R}$ *if the nonhomogeneous system (2) has exactly one bounded on* $\mathbb{R}$ *solution for each continuous and bounded on* $\mathbb{R}$ *function* $f(t)$*.*

Definition 2. *The system (1) is called weak-regular on* $\mathbb{R}$ *if the nonhomogeneous system (2) has at least one boundend on* $\mathbb{R}$ *solution for each continuous and bounded on* $\mathbb{R}$ *function* $f(t)$*.*

Definition 3. *If there exist a* $n \times n$*-dimensional matrix* $C(\tau) \in C^0(\mathbb{R})$ *such that the function of the form*

$$G(t,\tau) = \begin{cases} \Omega_\tau^t C(\tau), & \tau \le t, \\ \Omega_\tau^t\,[C(\tau) - I_n], & \tau > t, \end{cases} \tag{3}$$

satisfies the estimate

$$\|G(t,\tau)\| \le K \cdot exp\left\{-\gamma|t-\tau|\right\}, \tag{4}$$

where K *and* γ *are positive constants then function (3) is called the Green function of the problem of bounded solutions of system (1).*

In the case when there exists exactly one Green function, then the matrix $C(\tau)$ has the following property [6]:

$$C^2(\tau) \equiv C(\tau).$$

The existence of the Green function for system (1) allows us to claim that system (2) has a bounded on $\mathbb{R}$ solution for each $f(t) \in C^0(\mathbb{R})$. The solution can be written in integral form

$$x(t) = \int_{-\infty}^{+\infty} G(t,\tau) f(\tau) d\tau. \tag{5}$$

Remark 4. The existence of exactly one Green function (3) is equivalent to the regularity of system (1).

It is known [2, 3] that if there exists a quadratic form $V = \langle S(t)x, x\rangle$ with a symmetric matrix $S(t) \in C^1(\mathbb{R})$ such that the derivative related to the system (1) satisfies the inequality

$$\dot{V} = \left\langle \left[\dot{S(t)} + S(t)A(t) + A^T(t)S(t)\right] x, x\right\rangle \geq \|x\|^2, \tag{6}$$

what means that the quadratic form is positively definite and also the matrix $S(t)$ fulfills the following assumption

$$\det S(t) \neq 0, \quad \forall\, t \in \mathbb{R}, \tag{7}$$

then the system (1) is regular. The quadratic form V is called the generalized Lyapunov function.

In case of the systems (1) with constant coefficients the following statement is valid.

Proposition 5. *The system (1) in which $A(t) \equiv A$ is regular if and only if for all eigenvalues of matrix the A we have*

$$\Re\,(\lambda_j) \neq 0.$$

With system (1) we also consider the conjugated system

$$\dot{y} = -A^T(t)y. \tag{8}$$

The study over the systems (1) and (8) [2] shows that if the quadratic form $V = \langle S(t)x, x\rangle$ satisfies the assumption (7) then the inverse matrix $S^{-1}(t)$ will be bounded on $\mathbb{R}$ and also the derivative of a quadratic form

$$W = -\alpha \left\langle S^{-1}(t)y, y\right\rangle = \left\langle \overline{S}(t)y, y\right\rangle, \tag{9}$$

where $\alpha = \text{const} > 0$, $y \in \mathbb{R}^n$, related to the conjugated system (8) will be positively definite

$$\dot{W} = \left\langle \left[\dot{\overline{S}}(t) - \overline{S}(t)A^T(t) - A(t)\overline{S}(t)\right] y, y\right\rangle \geq \|y\|^2. \tag{10}$$

The following statement between those two systems takes place [2, 3].

Proposition 6. *If we assume that there exist a quadratic form (9) with a symmetric matrix* $\overline{S}(t) \in C^1(\mathbb{R})$ *(where* $\det \overline{S}(t)$ *can be equal to zero or not) for which the inequality (10) is satisfied then the system (1) will be weak-regular. When the condition* $\det \overline{S}(t) \neq 0, \forall t \in \mathbb{R}$ *is satisfied then the systems (1) and (8) will be regular on* $\mathbb{R}$*.*

It is known that each system (1) which is weak-regular on $\mathbb{R}$ can be transformed into regular system in the following way [3]:

$$\begin{cases} \dot{x} = A(t)x, \\ \dot{y} = x - A^T(t)y, \end{cases} \tag{11}$$

where the derivative related to the system (11) of the a quadratic form

$$\lambda \langle x, y \rangle + \left\langle \overline{S}(t)y, y \right\rangle, \tag{12}$$

will be positively definite for sufficiently large values of the parameter λ (at $\lambda > \|\overline{S}\|_0^2$. This proces is called a complementation of linear system to regular system. Let us notice that the matrix of the quadratic form (12) is nondegenerate matrix of the form

$$\begin{pmatrix} 0 & \frac{\lambda}{2} I_n \\ \frac{\lambda}{2} I_n & \overline{S}(t) \end{pmatrix}. \tag{13}$$

2. Main results

The main aim of this study was usage of the complementation process of linear system to regular system in study of the regularity of the system (1). The study over the complementation of linear systems to regular systems in case of linear extensions were discussed in more details in [1, 4].

Let consider the system which is a generalization of the system (11):

$$\begin{cases} \dot{x}_1 = A_{11}(t)x_1 + A_{12}(t)x_2, \\ \dot{x}_2 = A_{21}(t)x_1 + A_{22}(t)x_2, \end{cases} \tag{14}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$.

First we will analyse the case when $A_{11}(t) = A(t)$, $A_{22}(t) = -A^T(t)$ and $x_1 = x, x_2 = y$. Then the system (14) will have a form

$$\begin{cases} \dot{x} = A(t), \\ \dot{y} = A_{21}(t)x - A^T(t)y. \end{cases} \tag{15}$$

Let us assume that the continuous and bounded on $\mathbb{R}$ matrix $A_{21}(t)$ satisfies an inequality $\langle A_{21}x, x\rangle \geq \alpha\|x\|^2$, $\alpha = \text{const} > 0$. With this assumptions the system (15) will be regular because the derivative of a quadratic form (12) will be positively definite with sufficiently large values of the parameter λ.

Now let us analyse the general case of system (14). Let assume that for the subsystem of form

$$\dot{x}_2 = A_{22}(t)x_2, \quad x_2 \in \mathbb{R}^{n_2}, \tag{16}$$

there exists a quadratic form $\langle S_{22}(t)x_2, x_2\rangle$, $S_{22}(t) \in C_1(\mathbb{R})$ which derivative related to the system (16) is positively definite

$$\left\langle \left[\dot{S}_{22}(t) + S_{22}(t)A(t) + A^T(t)S_{22}(t)\right] x_2, x_2 \right\rangle \geq \|x_2\|^2. \tag{17}$$

In addition to this, let us assume that for all system (14) exists a quadratic form $\langle S(t)x, x\rangle$[1], where $S(t) \in C^1(\mathbb{R})$, which the derivative related to the system (14) satisfies an inequality

$$\left\langle \left[\dot{S}(t) + S(t)A(t) + A^T(t)S(t)\right] x, x \right\rangle \geq \|x_1\|^2. \tag{18}$$

With this assumptions in analogy to (11), if we consider the sum of quadratic forms

$$\lambda \langle S(t)x, x\rangle + \langle S_{22}(t)x_2, x_2\rangle, \tag{19}$$

we will see that its derivative related to the system (14) with sufficiently large values of the parameter $\lambda > 0$ is positively definite. By using the following significations

$$S_2(t) = \begin{pmatrix} 0 & 0 \\ 0 & S_{22}(t) \end{pmatrix}, \quad C_1 = \begin{pmatrix} I_{n_1} & 0 \\ 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 \\ 0 & I_{n_2} \end{pmatrix}, \tag{20}$$

[1] $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

the inequalities (17) and (18) can be rewritten as

$$\left\langle \left[\dot{S}_2(t) + S_2(t)A(t) + A^T(t)S_2(t)\right] C_2x, C_2x\right\rangle \geq \|C_2x\|^2, \tag{21}$$

$$\left\langle \left[\dot{S}(t) + S(t)A(t) + A^T(t)S(t)\right] (C_1 + C_2), (C_1 + C_2)\right\rangle \geq \|C_1x\|^2. \tag{22}$$

Now let in (21) and (22) the matrices S_2, C_1, C_2 do not must takes a form (20), but they can have the form $C_j = C_j(t)$, $j = 1, 2$. With this assumptions the following theorem takes place.

Theorem 7. *Let there exists the $n \times n$-dimensional symmetric matrices $S_j(t) \in C^1(\mathbb{R})$, $j = 1, 2$, for which the following inequalities are satisfied*

$$\begin{aligned} &\left\langle \left[\dot{S}_1(t) + S_1(t)A(t) + A^T(t)S_1(t)\right] (C_1(t) + C_2(t))\, x, \right. \\ &\left. (C_1(t) + C_2(t))\, x\right\rangle \geq \|C_1(t)x\|^2, \end{aligned} \tag{23}$$

$$\left\langle \left[\dot{S}_2(t) + S_2(t)A(t) + A^T(t)S_2(t)\right] C_2(t)x, C_2(t)x\right\rangle \geq \|C_2(t)x\|^2 \tag{24}$$

with the $n \times n$-dimensional matrices $C_j = C_j(t)$, $j = 1, 2$, which sum is non-degenerated matrix, i.e. $\det [C_1(t) + C_2(t)] \neq 0$, $\forall t \in \mathbb{R}$, *then the derivative of a quadratic form*

$$V_\lambda = \lambda \langle S_1(t)x, x\rangle + \langle S_2(t)x, x\rangle, \tag{25}$$

related to system (14), with sufficiently large values of the parameter λ, will be positively definite.

Proof. Let us write the derivative of a quadratic form (25) related to the system (14):

$$\begin{aligned} \dot{V}_\lambda = \lambda &\left\langle \left[\dot{S}_{[1}(t) + S_1(t)A(t) + A^T(t)S_1(t)\right] x, x\right\rangle + \\ &+ \left\langle \left[\dot{S}_2(t) + S_2(t)A(t) + A^T(t)S_2(t)\right] x, x\right\rangle = \lambda N_1 + N_2, \end{aligned}$$

where

$$\begin{aligned} N_1 = &\left\langle \left[\dot{S}_1(t) + S_1(t)A(t) + A^T(t)S_1(t)\right] x, x\right\rangle = \left\langle \left[\dot{S}_1(t) + S_1(t)A(t) + \right.\right. \\ &\left.\left. + A^T(t)S_1(t)\right] (C_1(t) + C_2(t))\, C^{-1}(t)x, (C_1(t) + C_2(t))\, C^{-1}(t)x\right\rangle. \end{aligned}$$

Denoting $C^{-1}(t)x = y$ and based on(23) we get

$$N_1 = \Big\langle \Big[\dot{S}_1(t) + S_1(t)A(t) + A^T(t)S_1(t)\Big] \left(C_1(t) + C_2(t)\right) y, \left(C_1(t) + C_2(t)\right) y \Big\rangle \geq \\ \geq \|C_1(t)y\|^2.$$

For the second component of the sum we have

$$\begin{aligned} N_2 = \Big\langle \Big[\dot{S}_2(t) + S_2(t)A(t) + A^T(t)S_2(t)\Big] \left(C_1(t) + C_2(t)\right) C^{-1}(t)x, \\ \left(C_1(t) + C_2(t)\right) C^{-1}(t)x \Big\rangle = \\ = \Big\langle \Big[\dot{S}_2(t) + S_2(t)A(t) + A^T(t)S_2(t)\Big] \left(C_1(t) + C_2(t)\right) y, \left(C_1(t) + C_2(t)\right) y \Big\rangle = \\ = \Big\langle \Big[\dot{S}_2(t) + S_2(t)A(t) + A^T(t)S_2(t)\Big] C_1(t)y, C_1(t)y \Big\rangle + \\ + 2\Big\langle \Big[\dot{S}_2(t) + S_2(t)A(t) + A^T(t)S_2(t)\Big] C_1(t)y, C_2(t)y \Big\rangle + \\ + \Big\langle \Big[\dot{S}_2(t) + S_2(t)A(t) + A^T(t)S_2(t)\Big] C_2(t)y, C_2(t)y \Big\rangle \geq \\ \geq -L\|C_1(t)y\|^2 - 2L\|C_1(t)y\|\|C_2(t)y\| + \|C_2(t)y\|^2. \end{aligned}$$

Based on the previous inequality we obtain

$$\begin{aligned} \dot{V}_\lambda \geq (\lambda - L)\|C_1(t)y\|^2 - 2L\|C_1(t)y\|\|C_2(t)y\| + \|C_2(t)y\|^2 \geq \\ \geq \frac{\lambda - L - L^2}{\lambda - L + 1}\left(\|C_1(t)y\|^2 + \|C_2(t)y\|^2\right) \geq \\ \geq \frac{\lambda - L - L^2}{2\left(\lambda - L + 1\right)}\|\left(C_1(t) + C_2(t)\right) y\|^2 = \frac{\lambda - L - L^2}{2\left(\lambda - L + 1\right)}\|x\|^2, \end{aligned}$$

$$\lambda - L - L^2 > 0.$$

The constant L is determined from the inequality:

$$\max_{\|x\|=1} \left|\Big\langle \Big[\dot{S}_j(t) + S_j(t)A(t) + A^T(t)S_j(t)\Big] x, x \Big\rangle\right| \leq L, \quad j = 1, 2.$$

□

Remark 8. The assumption that the matrices $C_1(t), C_2(t)$ and $\left(C_1(t) + C_2(t)\right)^{-1}$ are bounded on $\mathbb{R}$ is not necessary.

Now let assume that the system (1) can be written in the following form

$$\begin{cases} \dot{x}_1 = A_{11}(t)x_1 + A_{12}(t)x_2 + A_{13}(t)x_3, \\ \dot{x}_2 = A_{21}(t)x_1 + A_{22}(t)x_2 + A_{23}(t)x_3, \\ \dot{x}_3 = A_{31}(t)x_1 + A_{23}(t)x_2 + A_{33}(t)x_3, \end{cases} \tag{26}$$

where $x_j \in \mathbb{R}^{n_j}$, $j = 1, 2, 3$. Let us choose the subsystem of the system (26):

$$x_3 = A_{33}(t)x_3, \tag{27}$$

and say that for this subsystem there exists a quadratic form $V_3 = \langle S_{33}(t)x_3, x_3\rangle$, $S_{33}(t) \in C^1(\mathbb{R}) \cap C^0(\mathbb{R})$, which derivative related to the system (27) is positively definite

$$\dot{V}_3 = \left\langle \left[\dot{S}_{33}(t) + S_{33}(t)A_{33}(t) + A_{33}^T(t)S_{33}(t)\right] x_3, x_3 \right\rangle \geq \|x_3\|^3. \tag{28}$$

Now from the system (26) let us exclude the following subsystem

$$\begin{cases} \dot{x}_2 = A_{22}(t)x_2 + A_{23}(t)x_3, \\ \dot{x}_3 = A_{32}(t)x_2 + A_{33}(t)x_3. \end{cases} \tag{29}$$

To simplify the notation, let us denote

$$\tilde{A}(t) = \begin{pmatrix} A_{22}(t) & A_{23}(t) \\ A_{32}(t) & A_{33}(t) \end{pmatrix}, \quad \tilde{x} = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}.$$

Let us also assume that there exists a quadratic form $\tilde{V} = \left\langle \tilde{S}(t)\tilde{x}, \tilde{x} \right\rangle$, $\tilde{S}(t) \in C^1(\mathbb{R}) \cap C^0(\mathbb{R})$, which derivative related to the system (29) satisfies an inequality

$$\dot{\tilde{V}} = \left\langle \left[\dot{\tilde{S}}(t) + A(t)\tilde{S}(t) + A^T(t)\tilde{S}(t)\right] \tilde{x}, \tilde{x} \right\rangle \geq \|x_2\|^2. \tag{30}$$

Finally let assume that for all equations of system (26) there exists a quadratic form $V = \langle S_1(t)x, x\rangle$,[2] $S_1(t) \in C^1(\mathbb{R}) \cap C^0(\mathbb{R})$, which derivative related to the system (26) satisfies an inequality

$$\left\langle \left[\dot{S}_1(t) + S_1(t)A(t) + A^T(t)S_1(t)\right] x, x \right\rangle \geq \|x_1\|^2. \tag{31}$$

[2] $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

When all three inequalities (28), (30), (31) are satisfies then the derivative of the quadratic form

$$V_\lambda = \lambda_1 \langle S_1(t)x, x\rangle + \lambda_2 \left\langle \tilde{S}(t)\tilde{x}, \tilde{x}\right\rangle + \lambda_3 \langle S_{33}(t)x_3, x_3\rangle, \tag{32}$$

related to the system (26), with sufficiently large values of the parameters λ_j, $j = 1, 2, 3$, will be positively definite, i.e. $\dot{V}_\lambda \geq \|x\|^2$. By denoting

$$\begin{aligned} &C_3 = \operatorname{diag}\{I_{n_1}, 0, 0\}, \quad C_2 = \operatorname{diag}\{0, I_{n_2}, 0\}, \quad C_3 = \operatorname{diag}\{0, 0, I_{n_3}\}, \\ &S_2(t) = \operatorname{diag}\left\{0, \tilde{S}(t)\right\}, \quad S_3(t) = \operatorname{diag}\{0, 0, S_{33}(t)\}, \end{aligned} \tag{33}$$

the inequalities (28),(30),(31) can be written in the forms

$$\begin{aligned} &\left\langle \left[\dot{S}_1(t) + S_1(t)A(t)A^T(t)S_1(t)\right](C_1+C_2+C_3)\,x, (C_1+C_2+C_3)\,x\right\rangle \geq \|C_1x\|^2, \\ &\left\langle \left[\dot{S}_2(t) + S_2(t)A(t)A^T(t)S_2(t)\right](C_2+C_3)\,x, (C_2+C_3)\,x\right\rangle \geq \|C_2x\|^2, \\ &\left\langle \left[\dot{S}_3(t) + S_3(t)A(t)A^T(t)S_3(t)\right] C_3x, C_3x\right\rangle \geq \|C_3x\|^2. \end{aligned}$$

The following theorem takes place.

Theorem 9. *Let there exists the $n \times n$-dimensional symmetric matrices $S_j(t) \in C^1(\mathbb{R})$, $j = 1, 2, 3$, continuously differentiable and bounded on $\mathbb{R}$, for which are satisfied the following inequalities*

$$\begin{aligned} &\Big\langle \left[\dot{S}_1(t) + S_1(t)A(t)A^T(t)S_1(t)\right](C_1(t) + C_2(t) + C_3(t))\,x, \\ &\qquad (C_1(t) + C_2(t) + C_3(t))\,x\Big\rangle \geq \|C_1(t)x\|^2, \\ &\Big\langle \left[\dot{S}_2(t)+S_2(t)A(t)A^T(t)S_2(t)\right](C_2(t)+C_3(t))\,x, \\ &\qquad (C_2(t)+C_3(t))\,x\Big\rangle \geq \|C_2(t)x\|^2, \\ &\left\langle \left[\dot{S}_3(t) + S_3(t)A(t)A^T(t)S_3(t)\right] C_3(t)x, C_3(t)x\right\rangle \geq \|C_3(t)x\|^2, \end{aligned} \tag{34}$$

with the $n\times n$-dimensional matrices $C_j = C_j(t)$, $j = 1, 2, 3$, which sum is a nondegenerated matrix, i.e. $\det[C_1(t) + C_2(t) + C_3(t)] \neq 0$, $\forall t \in \mathbb{R}$. *Then the derivative of the quadratic form*

$$V_\lambda = \lambda_1 \langle S_1(t)x, x\rangle + \lambda_2 \langle S_2(t)x, x\rangle + \lambda_3 \langle S_3(t)x, x\rangle, \tag{35}$$

related to the system (26) will be positively definite with sufficiently large values of the parameters λ_j, $j = 1, 2, 3$.

Proof. Let us consider the sum of two symmetric matrices with positive parameter $\lambda > 0$:

$$S(t;\lambda) = \lambda S_2(t) + S_3(t). \tag{36}$$

We show that for sufficiently large values of the parameter $\lambda > 0$, the following inequality will be satisfied

$$\left\langle \left[\dot{S}(t;\lambda) + S(t;\lambda)A(t) + A^T(t)S(t;\lambda)\right](C_2(t) + C_3(t))\,x, (C_2(t) + C_3(t))\,x\right\rangle \geq$$
$$\geq \gamma(\lambda)\|C_2(t) + C_3(t)\|^2, \quad \gamma(\lambda) > 0. \tag{37}$$

Taking into account the linearity of matrix (36) related to parameter λ, the left hand side of (37) can be written as

$$\lambda\left\langle\left[\dot{S}_2(t) + S_2(t)A(t) + A^T(t)S_2(t)\right](C_2(t) + C_3(t))\,x, (C_2(t) + C_3(t))\,x\right\rangle +$$
$$+\left\langle\left[\dot{S}_3(t) + S_3(t)A(t) + A^T(t)S_3(t)\right](C_2(t) + C_3(t))\,x, (C_2(t) + C_3(t))\,x\right\rangle \geq$$
$$\geq \lambda\|C_2(t)x\|^2 + \left\langle\left[\dot{S}_3(t) + S_3(t)A(t) + A^T(t)S_3(t)\right]C_2(t)x, C_2(t)x\right\rangle +$$
$$+\,2\left\langle\left[\dot{S}_3(t) + S_3(t)A(t) + A^T(t)S_3(t)\right]C_2(t)x, C_3(t)x\right\rangle +$$
$$+\left\langle\left[\dot{S}_3(t) + S_3(t)A(t) + A^T(t)S_3(t)\right]C_3(t)x, C_3(t)x\right\rangle \geq$$
$$\geq (\lambda - L)\,\|C_2(t)x\|^2 - 2L\|C_2(t)x\|\|C_3(t)x\| + \|C_3(t)x\|^2 \geq$$
$$\geq \frac{\lambda - L - L^2}{2\,(\lambda - L + 1)}\|\,[C_2(t) + C_3(t)]\,x\|^2,$$

$$\lambda - L - L^2 > 0.$$

The constant L is determined from the inequality

$$\max_{\|x\|=1}\left|\left\langle\left[\dot{S}_j(t) + S_j(t)A(t) + A^T(t)S_j(t)\right]x, x\right\rangle\right| \leq L, \quad j = 1, 2, 3.$$

□

Putting in previous proof $C_1(t)+C_2(t)+C_3(t) = M_1(t)$, $C_2(t)+C_3(t) = M_2(t)$, $C_3(t) = M_3(t)$, the inequalities (34) will have the forms

$$\left\langle\left[\dot{S}_1(t) + S_1(t)A(t)A^T(t)S_1(t)\right]M_1(t)x, M_1(t)x\right\rangle \geq \|\,(M_1(t) - M_2(t))\,x\|^2,$$
$$\left\langle\left[\dot{S}_2(t) + S_2(t)A(t)A^T(t)S_2(t)\right]M_2(t)x, M_2(t)x\right\rangle \geq \|\,(M_2(t) - M_3(t))\,x\|^2,$$
$$\left\langle\left[\dot{S}_3(t) + S_3(t)A(t)A^T(t)S_3(t)\right]M_3(t)x, M_3(t)x\right\rangle \geq \|M_3(t)x\|^2.$$

In general case we have the following theorem.

Theorem 10. *Let there exists the $n \times n$-dimensional symmetric matrices $S_j(t)$ $j = \overline{1,k}$, continuously differentiable and bounded on $\mathbb{R}$, for which, are satisfied the following inequalities*

$$\begin{aligned}
&\left\langle \left[\dot{S}_j(t) + S_j(t)A(t)A^T(t)S_j(t)\right] M_j(t)x, M_j(t)x \right\rangle \geq \|\left(M_j(t) - M_{j+1}(t)\right)x\|^2,\\
&j = \overline{1,(k-1)},\\
&\left\langle \left[\dot{S}_k(t) + S_k(t)A(t)A^T(t)S_k(t)\right] M_k(t)x, M_k(t)x \right\rangle \geq \|M_k(t)x\|^2,
\end{aligned} \tag{38}$$

with the $n \times n$-dimensional matrices $M_j(t)$, $\det M_1(t) \neq 0$ $\forall t \in \mathbb{R}$, then the derivative of the quadratic form

$$V_\lambda = \sum_{j=1}^{n} \lambda_j \left\langle S_j(t)x, x \right\rangle, \tag{39}$$

related to the system (1) will be positively definite with sufficiently large values of the parameters $\lambda_j > 0$, $j = \overline{1,n}$.

Proof. First let us consider the sum of matrices with one parameter $\lambda_{k-1} > 0$:

$$\lambda_{k-1} S_{k-1}(t) + S_k(t) = \tilde{S}(t). \tag{40}$$

We show that for sufficiently large values of the parameter λ_{k-1} the following inequality will be satisfied

$$\begin{aligned}
&\left\langle \left[\dot{\tilde{S}}(t) + \tilde{S}(t)A(t) + A^T(t)\tilde{S}(t)\right] M_{k-1}(t)x, M_{k-1}(t)x \right\rangle \geq \\
&\qquad \geq \mu(\lambda_{k-1}) \cdot \|M_{k-1}(t)x\|^2,
\end{aligned} \tag{41}$$

where $\mu(\lambda_{k-1}) = \frac{\lambda_{k-1} - L - L^2}{2(\lambda_{k-1} - L + 1)}$, $\lambda_{k-1} > L + L^2$ and the constant L is determined by the inequality

$$\|\dot{S}_i + S_i(t)A(t) + A^T(t)S_i(t)\| \leq L, \quad i = \overline{1,k}. \tag{42}$$

Let us write the left hand side of (41) by substituting the sum (40) to the form

$$\lambda_{k-1}\left\langle\left[\dot{S}_{k-1}(t)+S_{k-1}(t)A(t)+A^T(t)S_{k-1}(t)\right]M_{k-1}(t)x,M_{k-1}(t)x\right\rangle+ \\ +\left\langle\left[\dot{S}_k(t)+S_k(t)A(t)+A^T(t)S_k(t)\right]\left(M_k(t)+\left(M_{k-1}(t)-M_k(t)\right)\right)x,\right. \\ \left.\left(M_k(t)+\left(M_{k-1}(t)-M_k(t)\right)\right)x\right\rangle. \quad (43)$$

The first component of the sum can by estimated in the following way

$$\lambda_{k-1}\left\langle\left[\dot{S}_{k-1}(t)+S_{k-1}(t)A(t)+A^T(t)S_{k-1}(t)\right]M_{k-1}(t)x,M_{k-1}(t)x\right\rangle\geq \\ \geq\|\left(M_{k-1}(t)-M_k(t)\right)x\|^2, \quad (44)$$

with positive constant λ_{k-1}. For the second component of the sum (43) we have

$$\left\langle\left[\dot{S}_k(t)+S_k(t)A(t)+A^T(t)S_k(t)\right]\left(M_k(t)+\left(M_{k-1}(t)-M_k(t)\right)\right)x,\left(M_k(t)+\right.\right. \\ \left.\left.+\left(M_{k-1}(t)-M_k(t)\right)\right)x\right\rangle= \\ =\left\langle\left[\dot{S}_k(t)+S_k(t)A(t)+A^T(t)S_k(t)\right]M_k(t)x,M_k(t)x\right\rangle+ \\ +2\left\langle\left[\dot{S}_k(t)+S_k(t)A(t)+A^T(t)S_k(t)\right]M_k(t)x,\left(M_{k-1}(t)-M_k(t)\right)x\right\rangle+ \\ +\left\langle\left[\dot{S}_k(t)+S_k(t)A(t)+A^T(t)S_k(t)\right]\left(M_{k-1}(t)-M_k(t)\right)x,\left(M_{k-1}(t)-M_k(t)\right)x\right\rangle.$$

We have the following estimates

$$2\left\langle\left[\dot{S}_k(t)+S_k(t)A(t)+A^T(t)S_k(t)\right]M_k(t)x,\left(M_{k-1}(t)-M_k(t)\right)x\right\rangle\geq \\ \geq-2L\|M_k(t)x\|\cdot\|\left(M_{k-1}(t)-M_k(t)\right)x\|, \quad (45)$$

$$2\left\langle\left[\dot{S}_k(t)+S_k(t)A(t)+A^T(t)S_k(t)\right]\left(M_{k-1}(t)-M_k(t)\right)x,\right. \\ \left.\left(M_{k-1}(t)-M_k(t)\right)x\right\rangle\geq-L\|\left(M_{k-1}(t)-M_k(t)\right)x\|^2. \quad (46)$$

From inequalities (44)-(46) we obtain

$$\left\langle\left[\dot{\tilde{S}}(t)+\tilde{S}(t)A(t)+A^T(t)\tilde{S}(t)\right]M_{k-1}(t)x,M_{k-1}(t)x\right\rangle\geq$$

$$\geq (\lambda_{k-1} - L) \left\| \left(M_{k-1}(t) - M_k(t)\right) x \right\|^2 - \\ - 2L \|M_k(t)x\| \cdot \left\| \left(M_{k-1}(t) - M_k(t)\right) x \right\| + \|M_k(t)x\|^2. \quad (47)$$

Denoting $\| (M_{k-1}(t) - M_k(t)) x \| = t_1$, $\|M_k(t)x\| = t_2$ the above considered quadratic form can be written in the form

$$\Phi(t_1, t_2) = (\lambda_{k-1} - L) t_1^2 - 2Lt_1t_2 + t_2^2,$$

which fulfills the estimate

$$\Phi(t_1, t_2) \geq \frac{\lambda_{k-1} - L - L^2}{\lambda_{k-1} - L + 1} \left(t_1^2 + t_2^2\right), \quad \lambda_{k-1} > L + L^2.$$

From this estimate and from inequality (47) we get

$$\left\langle \left[\dot{\tilde{S}}(t) + \tilde{S}(t)A(t) + A^T(t)\tilde{S}(t)\right] M_{k-1}(t)x, M_{k-1}(t)x \right\rangle \geq \\ \geq \frac{\lambda_{k-1} - L - L^2}{\lambda_{k-1} - L + 1} \left(\| (M_{k-1}(t) - M_k(t)) x\|^2 + \|M_k(t)x\|^2\right) \geq \\ \geq \frac{\lambda_{k-1} - L - L^2}{2(\lambda_{k-1} - L + 1)} \|M_k(t)x\|^2 = \mu(\lambda_{k-1}) \cdot \|M_{k-1}(t)x\|^2,$$

which justifies inequality (41).

Now let us consider the following sum of matrices

$$\lambda_{k-2}S_{k-2}(t) + \lambda_{k-1}(t) + Sk - 1(t) + S_k(t) = \lambda_{k-2}(t) + \tilde{S}(t) = \overline{S}(t). \quad (48)$$

We show that for sufficiently large values of the parameter λ_{k-2} the following inequality is satisfied

$$\left\langle \left[\dot{\overline{S}}(t) + \overline{S}(t)A(t) + A^T(t)\overline{S}(t)\right] M_{k-2}(t)x, M_{k-2}(t) \right\rangle \geq \\ \geq \mu(\lambda_{k-2}, \lambda_{k-1}) \|M_{k-2}(t)x\|^2, \quad (49)$$

where

$$\mu(\lambda_{k-2}, \lambda_{k-1}) = \frac{(\lambda_{k-2} - L(\lambda_{k-1} + 1))\,\mu(\lambda_{k-1}) - L^2 (\lambda_{k-1} + 1)^2}{2(\lambda_{k-2} - L(\lambda_{k-1} + 1) + \mu(\lambda_{k-1}))} > 0.$$

By rewriting the left hand side of an inequality (49) we obtain

$$\lambda_{k-2}\left\langle\left[\dot{S}_{k-2}(t)+S_{k-2}(t)A(t)+A^T(t)S_{k-2}(t)\right]M_{k-2}(t)x, M_{k-2}(t)x\right\rangle+$$
$$+\left\langle\left[\dot{\tilde{S}}(t)+\tilde{S}(t)A(t)+A^T(t)\tilde{S}(t)\right]M_{k-2}(t)x, M_{k-2}(t)x\right\rangle\geq$$
$$\geq\lambda_{k-2}\|\left(M_{k-2}(t)-M_{k-1}(t)\right)x\|^2+$$
$$+\left\langle\left[\dot{\tilde{S}}(t)+\tilde{S}(t)A(t)+A^T(t)\tilde{S}(t)\right]\left(M_{k-1}(t)+\left(M_{k-2}(t)-M_{k-1}(t)\right)\right)x,\right.$$
$$\left.\left(M_{k-1}(t)+\left(M_{k-2}(t)-M_{k-1}(t)\right)\right)x\right\rangle=$$
$$=\lambda_{k-2}\|\left(M_{k-2}(t)-M_{k-1}(t)\right)x\|^2+$$
$$+\left\langle\left[\dot{\tilde{S}}(t)+\tilde{S}(t)A(t)++A^T(t)\tilde{S}(t)\right]M_{k-1}(t)x, M_{k-1}(t)x\right\rangle+$$
$$+2\left\langle\left[\dot{\tilde{S}}(t)+\tilde{S}(t)A(t)+A^T(t)\tilde{S}(t)\right]M_{k-1}(t)x,\left(M_{k-2}(t)-M_{k-1}(t)\right)x\right\rangle+$$
$$+\left\langle\left[\dot{\tilde{S}}(t)+\tilde{S}(t)A(t)+A^T(t)\tilde{S}(t)\right]\left(M_{k-2}(t)-M_{k-1}(t)\right)x,\left(M_{k-2}(t)-M_{k-1}(t)\right)x\right\rangle.$$

Hence we get the following estimates

$$2\left\langle\left[\dot{\tilde{S}}(t)+\tilde{S}(t)A(t)+A^T(t)\tilde{S}(t)\right]M_{k-1}(t)x,\left(M_{k-2}(t)-M_{k-1}(t)\right)x\right\rangle\geq$$
$$\geq-2L\left(\lambda_{k-1}+1\right)\|M_{k-1}(t)x\|\cdot\|\left(M_{k-2}(t)-M_{k-1}(t)\right)x\|,$$

$$\left\langle\left[\dot{\tilde{S}}(t)+\tilde{S}(t)A(t)+A^T(t)\tilde{S}(t)\right]\left(M_{k-2}(t)-M_{k-1}(t)\right)x,\right.$$
$$\left.\left(M_{k-2}(t)-M_{k-1}(t)\right)x\right\rangle\geq-L\left(\lambda_{k-1}+1\right)\|\left(M_{k-2}(t)-M_{k-1}(t)\right)x\|^2,$$

from which we can evaluate the left side of the inequality (49):

$$\left\langle\left[\dot{\overline{S}}(t)+\overline{S}(t)A(t)+A^T(t)\overline{S}(t)\right]M_{k-2}(t)x, M_{k-2}(t)\right\rangle\geq$$
$$\geq\left(\lambda_{k-2}-\lambda_{k-1}L-L\right)\|\left(M_{k-2}(t)-M_{k-1}(t)\right)x\|^2-$$
$$-2L\left(\lambda_{k-1}+1\right)\|M_{k-1}(t)x\|\|\left(M_{k-2}(t)-M_{k-1}(t)\right)x\|+\mu\left(\lambda_{k-1}\right)\|M_{k-1}(t)x\|^2\geq$$
$$\geq\frac{\left(\lambda_{k-2}-L\left(\lambda_{k-1}+1\right)\right)\mu(\lambda_{k-1})-L^2\left(\lambda_{k-1}+1\right)^2}{\lambda_{k-2}-L\left(\lambda_{k-1}+1\right)+\mu(\lambda_{k-1})}$$
$$\left(\|\left(M_{k-2}(t)-M_{k-1}(t)\right)x\|^2+\|M_{k-1}(t)x\|^2\right)\geq$$
$$\geq\frac{\left(\lambda_{k-2}-L\left(\lambda_{k-1}+1\right)\right)\mu(\lambda_{k-1})-L^2\left(\lambda_{k-1}+1\right)^2}{2\left(\lambda_{k-2}-L\left(\lambda_{k-1}+1\right)+\mu(\lambda_{k-1})\right)}\|M_{k-2}(t)x\|^2.$$

This inequality proves inequality (49).

In analogy to the combination of symmetric matrices

$$S_\lambda(t) = \lambda_1 S_1(t) + \lambda_2 S_2(t) + \ldots + \lambda_{k-1} S_{k-1}(t) + S_k(t)$$

we get the following inequality

$$\left\langle \left[\dot{S}_\lambda(t) + S_\lambda(t) A(t) + A^T(t) S_\lambda \right] M_1(t)x, M_1(t)x \right\rangle \geq$$
$$\geq \mu\left(\lambda_1, \lambda_2, \ldots, \lambda_{k-2}, \lambda_{k-1}\right) \|M_1(t)x\|^2, \quad (50)$$

where the positive constant $\mu\left(\lambda_1, \lambda_2, \ldots, \lambda_{k-2}, \lambda_{k-1}\right)$ can be written in the form

$$\mu\left(\lambda_1, \lambda_2, \ldots, \lambda_{k-2}, \lambda_{k-1}\right) =$$
$$= \frac{\lambda_1 - L\left(1 + \lambda_2 + \ldots + \lambda_{k-1}\right)\mu\left(\lambda_2, \ldots, \lambda_{k-1}\right) - L^2\left(1 + \lambda_2 + \ldots + \lambda_{k-1}\right)^2}{2\left(\lambda_1 - L\left(1 + \lambda_2 + \ldots + \lambda_{k-1}\right) + \mu\left(\lambda_2, \ldots, \lambda_{k-1}\right)\right)}, \quad (51)$$

For the positive coefficients the following formula takes place

$$\mu\left(\lambda_j, \lambda_{j+1}, \ldots, \lambda_{k-1}\right) =$$
$$= \frac{\left(\lambda_j - L\left(1 + \lambda_{j+1} + \ldots + \lambda_{k-1}\right)\right)\mu\left(\lambda_{j+1}, \ldots, \lambda_{k-1}\right) - L^2\left(1 + \lambda_{j+1} + \ldots + \lambda_{k-1}\right)}{2\left(\lambda_1 - L\left(1 + \lambda_{j+1} + \ldots + \lambda_{k-1}\right) + \mu\left(\lambda_{j+1}, \ldots, \lambda_{k-1}\right)\right)} \quad (52)$$

where for $j = k-2$ takes place the following equality $\mu\left(\lambda_{k-1}\right) = \frac{\lambda_{k-1} - L - \lambda_2}{2(\lambda_{k-1} - \lambda + 1)}$, $\lambda_{k-1} > L + L^2$. The constant L is determined by the inequality (42). Because in inequality (50) the matrix $M_1(t)$ is nondegenerated so it implies that the derivative of the quadratic form (39), related to the system (1), will be positively definite for sufficiently large values of the parameters $\lambda_1, \lambda_2, \ldots, \lambda_{k-1} > 0$. $\square$

Remark 11. From the formulas (51) and (52) follows that, for sufficiently large values of the parameters $\lambda_{k-2}, \lambda_{k-1}$ the coefficient $\mu\left(\lambda_{k-2}, \lambda_{k-1}\right)$ takes the value nearly to $\frac{1}{4}$.

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