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# SOLUTION OF SOME KRONHEIMER'S PROBLEM ON SUMMABLE SEQUENCES

**Summary**. In the present paper we give the solution of E. Kronheimer's problem (problem A6516 in Amer. Math. Month.), alternative to three other solutions included in paper [1].

# ROZWIĄZANIE PROBLEMU KRONHEIMERA O CIĄGACH SUMOWALNYCH

**Streszczenie**. W artykule przedstawiono rozwiązanie problemu E. Kronheimera (problem A6515 z Amer. Math. Monthly) alternatywne do trzech innych rozwiązań tego problemu zawartych w pracy [1].

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## Problem

We will construct an example of sequence $\{s_n\}_{n=1}^{\infty}$ of real numbers such that $s_n \neq 0$ for infinitely many subscripts $n \in \mathbb{N}$ and such that each of the following series is convergent

$$s_1 + s_2 + s_3 + \ldots,$$

$$s_1 + (s_1 + s_2) + (s_1 + s_2 + s_3) + \ldots,$$

$$s_1 + [s_1 + (s_1 + s_2)] + [s_1 + (s_1 + s_2) + (s_1 + s_2 + s_3)] + \ldots$$

Moreover, on the basis of presented construction of sequence $\{s_n\}$, for each $r \in \mathbb{N}$ we give the example of sequence $\{\sigma_r(1,n)\}_{n\geqslant 1}$ such that the respective series $\sum\limits_{n\geqslant 1} \sigma_r(p,n)$, $p \in \mathbb{N}$, where $\sigma_r(k+1,n) = \sum\limits_{i=1}^{n} \sigma_r(k,i)$, $k, n \in \mathbb{N}$, are convergent if and only if $p \leqslant r$.

## Solution

Let us introduce the following notation

$$s(1,n) = s_n, \qquad n \in \mathbb{N},$$

$$s(k+1,n) = \sum_{i=1}^{n} s(k,i), \qquad k, n \in N,$$

where sequence $\{s(1,n)\}_{n\geqslant 1}$ will be constructed in the successive steps of the following algorithm. Thus, in the first step of this algorithm we set

$$s(1,1) = \frac{1}{2},$$

$$s(1,2) = -\frac{1}{2}.$$

In the second step we take

$$\begin{array}{ll} s(1,3) = -\frac{1}{4}, & s(1,4) = \frac{1}{4}, \\ s(1,5) = -\frac{1}{4}, & s(1,6) = \frac{1}{4}. \end{array}$$

Then we have

$$\begin{array}{ll} s(2,1) = \frac{1}{2}, & s(2,2) = 0, \\ s(2,3) = -\frac{1}{4}, & s(2,4) = 0, \\ s(2,5) = -\frac{1}{4}, & s(2,6) = 0. \end{array}$$

Now, let us assume that after the $k$-th step of the algorithm we have defined the elements $s(1,n)$, for $1 \leqslant n \leqslant m(k)$, where $m(k)$ is some natural number. Moreover, we assume that the following equalities are satisfied

$$\sum_{n=1}^{m(k)} s(p,n) = 0 \qquad \text{for each } 1 \leqslant p \leqslant k.$$

Let us notice that $m(1) = 2$ and $m(2) = 6$ and

$$\sum_{n=1}^{m(1)} s(1,n) = \frac{1}{2} - \frac{1}{2} = 0,$$

$$\sum_{n=1}^{m(2)} s(1,n) = \frac{1}{2} - \frac{1}{2} - \frac{1}{4} + \frac{1}{4} - \frac{1}{4} + \frac{1}{4} = 0,$$

$$\sum_{n=1}^{m(2)} s(2,n) = \frac{1}{2} - \frac{1}{4} - \frac{1}{4} = 0.$$

Now we proceed to description of $(k+1)$ step of the algorithm. Let us put

$$s(1, p \cdot m(k) + n) = -(k+1)^{-1} s(1,n) \tag{1}$$

for any $n \in \mathbb{N}$, $1 \leqslant p \leqslant k+1$, $1 \leqslant n \leqslant m(k)$. Let us notice that then, with respect to (1), the following relations hold true

$$|s(p,n)| \leqslant (k+1)^{-1} \tag{2}$$

for every $p, n \in \mathbb{N}$, $1 \leqslant p \leqslant k$, $m(k) < n \leqslant m(k+1)$, where

$$m(k+1) := (k+2)m(k)$$

and

$$\sum_{n=1}^{m(k+1)} s(p,n) = \sum_{n=1}^{m(k)} s(p,n) + (k+1) \left[ -(k+1)^{-1} \sum_{n=1}^{m(k)} s(p,n) \right] = 0 \tag{3}$$

for every $p \in \mathbb{N}$, $1 \leqslant p \leqslant k+1$. Certainly, identities (3) ensure correctness of this construction. For example, in the third step of our procedure we define

$$\begin{array}{ll} s(1,7) = -\frac{1}{6}, & s(1,8) = \frac{1}{6}, \\ s(1,9) = \frac{1}{12}, & s(1,10) = -\frac{1}{12}, \\ s(1,11) = \frac{1}{12}, & s(1,12) = -\frac{1}{12}, \\ s(1,13) = -\frac{1}{6}, & s(1,14) = \frac{1}{6}, \end{array}$$

$$\begin{array}{ll} s(1,15)=\frac{1}{12}, & s(1,16)=-\frac{1}{12},\\ s(1,17)=\frac{1}{12}, & s(1,18)=-\frac{1}{12},\\ s(1,19)=-\frac{1}{6}, & s(1,20)=\frac{1}{6},\\ s(1,21)=\frac{1}{12}, & s(1,22)=-\frac{1}{12},\\ s(1,23)=\frac{1}{12}, & s(1,24)=-\frac{1}{12}. \end{array}$$

Then we get

$$\begin{array}{ll} s(2,7)=-\frac{1}{6}, & s(2,8)=0,\\ s(2,9)=\frac{1}{12}, & s(2,10)=0,\\ s(2,11)=\frac{1}{12}, & s(2,12)=0,\\ s(2,13)=-\frac{1}{6}, & s(2,14)=0,\\ s(2,15)=\frac{1}{12}, & s(2,16)=0,\\ s(2,17)=\frac{1}{12}, & s(2,18)=0,\\ s(2,19)=-\frac{1}{6}, & s(2,20)=0,\\ s(2,21)=\frac{1}{12}, & s(2,22)=0,\\ s(2,23)=\frac{1}{12}, & s(2,24)=0 \end{array}$$

and

$$\begin{array}{ll} s(3,1)=\frac{1}{2}, & s(3,2)=\frac{1}{2},\\ s(3,3)=\frac{1}{4}, & s(3,4)=\frac{1}{4},\\ s(3,5)=0, & s(3,6)=0,\\ s(3,7)=-\frac{1}{6}, & s(3,8)=-\frac{1}{6},\\ s(3,9)=-\frac{1}{12}, & s(3,10)=-\frac{1}{12},\\ s(3,11)=0 & s(3,12)=0,\\ s(3,13)=-\frac{1}{6}, & s(3,14)=-\frac{1}{6},\\ s(3,15)=-\frac{1}{12}, & s(3,16)=-\frac{1}{12},\\ s(3,17)=0, & s(3,18)=0,\\ s(3,19)=-\frac{1}{6}, & s(3,20)=-\frac{1}{6},\\ s(3,21)=-\frac{1}{12}, & s(3,22)=-\frac{1}{12},\\ s(3,23)=0, & s(3,24)=0. \end{array}$$

With regard to the Principle of Mathematical Induction the sequence $\{s(1,n)\}_{n\geqslant 1}$ is defined. From (2) we obtain, in particular, that

$$|s(p,n)| \leqslant (k+1)^{-1}$$

for every $p, n \in \mathbb{N}$, $1 \leqslant p \leqslant k$ and $n > m(k)$, which implies the convergence of every series $\sum\limits_{n\geqslant 1} s(p,n)$, $p \in \mathbb{N}$. Moreover, let us notice that not all the elements of sequence $\{s(1,n)\}_{n\geqslant 1}$ are zeros. For example, from definition of $\{s(1,n)\}_{n\geqslant 1}$ we have that

$$s(1, p \cdot m(k) + i) = (-1)^i [2(k+1)]^{-1}$$

for $k \in \mathbb{N}$, $1 \leqslant p \leqslant k+1,\ i = 1, 2$.

## Construction of the sequence $\{\sigma_r(1,n)\}_{n=1}^{\infty}$

Presented here the method of construction can be called as the method of reconstruction. For this, let us fix $r \in \mathbb{N}$. We put

$$\sigma_r(r,n) = \begin{cases} k^{-1} & \text{for } n = 2k-1,\ k \in \mathbb{N}, \\ -k^{-1} & \text{for } n = 2k,\ k \in \mathbb{N}. \end{cases}$$

One can easily verify that

$$\sigma_r(r+1,n) = \begin{cases} k^{-1} & \text{for } n = 2k-1,\ k \in \mathbb{N}, \\ 0 & \text{for } n = 2k,\ k \in \mathbb{N}, \end{cases}$$

which implies the divergence of series $\sum\limits_{n\geqslant 1} \sigma_r(r+1,n)$ and, with respect to the non-negativity of elements of this series, also the divergence of series $\sum\limits_{n\geqslant 1} \sigma_r(p,n)$ for $p > r+1$. We set

$$\sigma_r(r-1,1) := \sigma_r(r,1),$$

$$\sigma_r(r-1,n) := \sigma_r(r,n) - \sigma_r(r,n-1) \qquad \text{for } n > 1.$$

Series $\sum\limits_{n\geqslant 1} \sigma_r(r-1,n)$ is convergent, since the sequence of its partial sums, it means the sequence $\{\sigma_r(r,n)\}_{n\geqslant 1}$, converges to zero. Reasoning in the similar way we reveal the convergence of series $\sum\limits_{n\geqslant 1} \sigma_r(i,n)$ defined for $1 \leqslant i \leqslant r-1$ in the following way

$$\sigma_r(i-1,1) = \sigma_r(i,1),$$

$$\sigma_r(i-1,n) = \sigma_r(i,n) - \sigma_r(i,n-1), \quad n > 1.$$

**Remark 1.** Presented above the method of constructing the sequences $\{\sigma_r(1,n)\}$, $r \in \mathbb{N}$, gives in each case the series with infinitely many elements different from zero (indeed, if any of the reconstructed series would have, starting from some term, only zeros then the convergent preceding series also possess this property, which is not true since, let us recall, we start with the following series

$$1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \dots, \tag{4}$$

all elements of which are different from zero).

If the assumption, that the series $\sum_{n\geqslant 1} \sigma_r(1,n)$ must have infinitely many elements different from zero, could be neglected then we could start, instead of series (4), with the following “not much interesting” series

$$1+0+0+0+0+\ldots.$$

# References

1. Kronheimer E., Vitale R., Zwicker W., Pelling M., Willekens E.: *Infinitely summable sequences.* Amer. Math. Month. **94**, no. 10 (1987), 1014–1019.