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# SOME INTRIGUING LIMITS – CONTINUATION

**Summary**. Wituła and Słota in [College Math. J. **42** (2011), 328] proposed a way of proving the relation (1) given below which appeared to be a genuine result. Authors of the present paper, inspired by the form of this limit, try to find some generalizations of this one, also in the context of some special functions (e.g. the gamma function, the generalized Laguerre polynomials).

# O PEWNYCH INTRYGUJĄCYCH GRANICACH – KONTYNUACJA

**Streszczenie**. Wituła i Słota w notce [College Math. J. **42** (2011), 328] zaproponowali udowodnienie relacji (1), podanej poniżej, która wydaje się bardzo ciekawą zależnością. Autorzy niniejszego artykułu, zainspirowani postacią tej granicy, próbują znaleźć różne jej uogólnienia także w kontekście pewnych funkcji specjalnych (np. funkcji gamma, uogólnionych wielomianów Laguerre'a).

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Problem of evaluating the limits of functions depending on the expressions $(1+\frac{\alpha}{x})^x$ $(x \to \infty)$ is like never ending story. Look at the following ones (see [15]):

$$\lim_{x\to\infty}\Big(e^{\frac{(-1)^n}{n}}\Big(\ldots\Big(e^{\frac{1}{4}}\Big(e^{-\frac{1}{3}}\Big(e^{\frac{1}{2}}\Big(e^{-1}\Big(1+\frac{1}{x}\underbrace{\Big)^x\Big)^x\Big)^x\Big)^x\Big)^x\ldots\Big)^x}_{(n+1)-\text{times}} = e^{\frac{(-1)^n}{n+1}} \tag{1}$$

for every $n \in \mathbb{N}$;

$$\lim_{x\to\infty}\Big(\exp(\mathbb{A}_n)\Big(\ldots\Big(\exp(\mathbb{A}_2)\Big(\exp(\mathbb{A}_1)\frac{\prod_{i=1}^{r}\big(1+\frac{\alpha_i}{x}\big)^{\beta_i x}}{\prod_{j=1}^{r}\big(1+\frac{\gamma_j}{x}\big)^{\delta_j x}}\underbrace{\Big)^x\Big)^x\ldots\Big)^x\Big)^x}_{n-\text{times}} = \exp(-\mathbb{A}_{n+1}), \tag{2}$$

for every $n, r \in \mathbb{N}$, $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R}$, $i = 1, \ldots, r$ where

$$\mathbb{A}_k := \frac{1}{k}\begin{vmatrix} \boldsymbol{\alpha}_k & \boldsymbol{\delta} \\ \boldsymbol{\gamma}_k & \boldsymbol{\beta} \end{vmatrix} := \frac{1}{k}\left(\boldsymbol{\alpha}_k \circ \boldsymbol{\beta} - \boldsymbol{\gamma}_k \circ \boldsymbol{\delta}\right), \qquad k \in \mathbb{N},$$

and the symbol $\circ$ denotes the scalar product applied to the vectors $\boldsymbol{\alpha}_k, \boldsymbol{\beta}, \boldsymbol{\gamma}_k, \boldsymbol{\delta} \in \mathbb{R}^r$, defined in the following way

$$\boldsymbol{\alpha}_k := [(-\alpha_1)^k, (-\alpha_2)^k, \ldots, (-\alpha_r)^k], \qquad \boldsymbol{\beta} := [\beta_1, \beta_2, \ldots, \beta_r],$$
$$\boldsymbol{\gamma}_k := [(-\gamma_1)^k, (-\gamma_2)^k, \ldots, (-\gamma_r)^k], \qquad \boldsymbol{\delta} := [\delta_1, \delta_2, \ldots, \delta_r];$$

$$\lim_{x\to\infty}\left(\frac{2\,\alpha\,\beta\,x}{(\alpha-\beta)\,e}\left(\Big(1+\frac{1}{\alpha\,x}\Big)^{\alpha x} - \Big(1+\frac{1}{\beta\,x}\Big)^{\beta x}\right)\right)^x = \exp\left(-\frac{11}{12}\Big(\frac{1}{\alpha}+\frac{1}{\beta}\Big)\right) \tag{3}$$

for every $\alpha, \beta \in \mathbb{R}\setminus\{0\}$, $\alpha \neq \beta$;

$$\lim_{x\to\infty}\left(\ln\left(\ln\Big(1+\frac{1}{x}\Big)^x\right)^{-2x}\right)^x = \exp\Big(-\frac{5}{12}\Big); \tag{4}$$

$$\lim_{x\to\infty} x\left(e - \Big(1+\frac{1}{x}\Big)^x\right) = \frac{e}{2}; \tag{5}$$

$$\lim_{x\to\infty} x^2\left(e - \frac{e}{2x} - \Big(1+\frac{1}{x}\Big)^x\right) = -\frac{11}{24}e. \tag{6}$$

*Sketch of the proofs of (1) and (3).*
We have ($x > 1$):

$$\Big(1+\frac{1}{x}\Big)^x = \exp\Big(x\,\ln\Big(1+\frac{1}{x}\Big)\Big) = \exp\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)\,x^n}\right)$$

which implies (1). Now, let $\alpha, \beta \in \mathbb{R}\setminus\{0\}$, $\alpha \neq \beta$. Then we get

$$\begin{aligned}\Big(1+\frac{1}{\alpha x}\Big)^{\alpha x} - \Big(1+\frac{1}{\beta x}\Big)^{\beta x} &= \exp\Big(\alpha\,x\,\ln\Big(1+\frac{1}{\alpha x}\Big)\Big) - \exp\Big(\beta\,x\,\ln\Big(1+\frac{1}{\beta x}\Big)\Big) = \\ = \exp\Big(1-\frac{1}{2\alpha x}+\frac{1}{3\alpha^2x^2}+o\Big(\frac{1}{x^2}\Big)\Big) &- \exp\Big(1-\frac{1}{2\beta x}+\frac{1}{3\beta^2x^2}+o\Big(\frac{1}{x^2}\Big)\Big) = \\ &= \exp\Big(1-\frac{1}{2\beta x}+\frac{1}{3\beta^2x^2}+o\Big(\frac{1}{x^2}\Big)\Big)\times \\ &\times\Big[\exp\Big(\frac{1}{2\beta x}-\frac{1}{2\alpha x}+\frac{1}{3\alpha^2x^2}-\frac{1}{3\beta^2x^2}+o\Big(\frac{1}{x^2}\Big)\Big)-1\Big] = \\ &= e\,\exp\Big(-\frac{1}{2\beta x}+o\Big(\frac{1}{x}\Big)\Big)\Big[\frac{\alpha-\beta}{2\alpha\beta x}-\frac{(\alpha-\beta)(5\alpha+11\beta)}{24\alpha^2\beta^2x^2}+o\Big(\frac{1}{x^2}\Big)\Big],\end{aligned}$$

which implies

$$\begin{aligned}&\left(\frac{2\alpha\beta x}{(\alpha-\beta)e}\Big[\Big(1+\frac{1}{\alpha x}\Big)^{\alpha x}-\Big(1+\frac{1}{\beta x}\Big)^{\beta x}\Big]\right)^x = \\ &\qquad = \exp\Big(-\frac{1}{2\beta}+o(1)\Big)\Big(1-\frac{5\alpha+11\beta}{12\alpha\beta x}+o\Big(\frac{1}{x}\Big)\Big)^x,\end{aligned}$$

which easily proves (3). □

*Proof of (2).*
We have ($x > \max\{|\alpha|, |\beta|\}$):

$$\begin{aligned}\frac{\big(1+\frac{\alpha}{x}\big)^{\beta x}}{\big(1+\frac{\gamma}{x}\big)^{\delta x}} &= \exp\left(\beta\,x\,\ln\Big(1+\frac{\alpha}{x}\Big)-\delta\,x\,\ln\Big(1+\frac{\gamma}{x}\Big)\right) = \\ &= \exp\left(\sum_{n=0}^{\infty}(-1)^n\big(\beta\,\alpha^{n+1}-\delta\,\gamma^{n+1}\big)\,\frac{1}{(n+1)x^n}\right),\end{aligned}$$

which implies (2). □

*Sketch of the proofs of (5) and (6).*
By substitution $\frac{1}{x} = t$ we get

$$\lim_{x\to\infty} x\left(e-\Big(1+\frac{1}{x}\Big)^x\right) = \lim_{t\to 0^+}\frac{e-(1+t)^{\frac{1}{t}}}{t},$$

$$\lim_{x\to\infty} x^2\left(e-\frac{e}{2x}-\left(1+\frac{1}{x}\right)^x\right)=\lim_{t\to 0^+}\frac{e-\frac{e}{2}\cdot t-(1+t)^{\frac{1}{t}}}{t^2}.$$

Moreover we have ($x>1 \Rightarrow t\in(0,1)$):

$$(1+t)^{\frac{1}{t}}=\exp\Big(\frac{1}{t}\ln(1+t)\Big)=\exp\Big(\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}t^n\Big). \tag{7}$$

Application of l'Hospital's rule and relation (7) gives the values of both limits. □

*Proof of (4).*
We have

$$\ln\Big(1+\frac{1}{x}\Big)^x=1-\frac{1}{2x}+\frac{1}{3x^2}+o\Big(\frac{1}{x^2}\Big)$$

and

$$\ln\left(1-\frac{1}{2x}+\frac{1}{3x^2}+o\Big(\frac{1}{x^2}\Big)\right)^{-2x}=1-\frac{5}{12x}+o\Big(\frac{1}{x}\Big),$$

which implies (4), since

$$\lim_{x\to\infty}\Big(1-\frac{5}{12x}+o\Big(\frac{1}{x}\Big)\Big)^x=\exp\Big(-\frac{5}{12}\Big).$$

□

**Remark 1.** We note that limit (1) for $n=1$ is "a regular guest" of many problems in calculus books (see e.g [4,6,8]). In contrast, the limit of general shape (1) and its generalizations (2) are probably new.

**Remark 2.** Limits (3)–(6), which are probably new (see e.g. [4,6–8,10,11,15]), arose during the discussion on generalizations of limits (1) and (2) (and they are far from our expectations). Authors hope that these limits inspire the Readers to look for another results.

**Remark 3.** We note that

$$\lim_{x\to+\infty} x\left(\Big(1+\frac{1}{x-\alpha}\Big)^x-\Big(1+\frac{1}{x}\Big)^x\right)=\alpha\, e$$

for every $\alpha\in\mathbb{R}$.

*Proof.* From the following formula

$$\Big(1+\frac{a}{x}\Big)^x=\exp\Big(x\,\ln\Big(1+\frac{a}{x}\Big)\Big)=\exp(a)\,\exp\left(\sum_{n=1}^{\infty}\frac{(-1)^n a^{n+1}}{(n+1)\,x^n}\right)=$$

$$=e^a\left(\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{n=1}^{\infty}\frac{(-1)^n a^{n+1}}{(n+1)\,x^n}\right)^k\right)=$$

$$=e^a\left(1-\frac{a^2}{2\,x}+\Big(\frac{a}{8}+\frac{1}{3}\Big)\frac{a^3}{x^2}-\Big(\frac{a^2}{24}+\frac{a}{3}+\frac{1}{2}\Big)\frac{a^4}{2\,x^3}+\dots\right),$$

for $a=1$ and from the following decomposition (which is a special form of binomial series):

$$\Big(1+\frac{1}{x-\alpha}\Big)^{\alpha}=1+\frac{\alpha}{x-\alpha}+o\Big(\frac{1}{x-\alpha}\Big)$$

we get

$$x\left(\Big(1+\frac{1}{x-\alpha}\Big)^{x}-\Big(1+\frac{1}{x}\Big)^{x}\right)=$$

$$=\Big(1+\frac{1}{x-\alpha}\Big)^{\alpha}x\left(\Big(1+\frac{1}{x-\alpha}\Big)^{x-\alpha}-e\right)+x\left(e-\Big(1+\frac{1}{x}\Big)^{x}\right)+$$

$$+e\,x\left(\Big(1+\frac{1}{x-\alpha}\Big)^{\alpha}-1\right)\underset{x\to\infty}{\longrightarrow}-\frac{e}{2}+\frac{e}{2}+\alpha\,e=\alpha\,e.$$

□

**Remark 4.** In paper [5] the authors observed that from formula (1) for $n=1$ and from Stirling's formula (see e.g. [9]) the following approximation holds

$$\Gamma(x+1)\sim\sqrt{\frac{2\pi x}{e}}\cdot\frac{x^{x^2+x}}{(x+1)^{x^2}}.$$

Starting from this formula Feng and Wang (authors of [5]) proved that for sufficiently large $x\in\mathbb{R}$ the following one holds

$$\Gamma(x+1)=\sqrt{2\pi}\cdot x^{x+\frac{1}{2}}\Big(\frac{x-1}{x+1}\Big)^{\frac{x^2}{2}+\sum_{k=0}^{m}\frac{\alpha_k}{x^{2k}}+O\left(\frac{1}{x^{2m+2}}\right)},$$

where constants $\alpha_k$ satisfy the recurrence relation

$$\sum_{j=0}^{k}\frac{\alpha_j}{2k-2j+1}=-\frac{1}{2(2k+3)}-\frac{B_{2k+2}}{2(2k+1)(2k+2)}$$

for $k=0,1,2,\dots$ and $B_n$ denotes the $n$-th Bernoulli number (see e.g. [9]).

**Remark 5.** In paper [13] the asymptotic expansion of the Gamma functions ratio is obtained

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \sim z^{\alpha-\beta}\Bigg(1+(\alpha-\beta)(\alpha+\beta-1)\frac{1}{2z}+ \\ +\binom{\alpha-\beta}{2}\Big(3(\alpha+\beta-1)^2-\alpha+\beta-1\Big)\frac{1}{12z^2}+O\big(z^{-3}\big)\Bigg),$$

where $\alpha, \beta, z \in \mathbb{C}$ and $z \to \infty$. From this expansion we deduce the relation

$$\lim_{z\to\infty}\left(z^{\beta-\alpha}\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}\right)^z = \frac{1}{2}(\alpha-\beta)(\alpha+\beta-1),$$

where each power has its principal value.

**Remark 6.** In paper [14] the formulae of type (1) connected with limit

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = e^{-1}$$

are discussed. For example, there are presented the following relations

$$\lim_{n\to\infty}\left(e^{-\frac{1}{12}}\frac{\left(e\cdot\frac{\sqrt[n]{n!}}{n}\right)^n}{\sqrt{2\pi n}}\right)^n = 1$$

and

$$\lim_{n\to\infty}\left(\frac{n!}{\sqrt{2\pi}}\left(\frac{e}{\sqrt{n^2+n+\frac{1}{6}}}\right)^{n+\frac{1}{2}}\right)^{n^3} = e^{\frac{1}{240}}.$$

**Remark 7.** Let $L_n^{(-a)}(-z)$ denote the $n$-th generalized Laguerre polynomial (see [12]). Then we obtain the relation (see [1–3]):

$$\lim_{n\to\infty}\left(L_n^{(-a)}(-z)\cdot\left(\frac{e^{-\frac{z}{2}}}{2\sqrt{\pi}}\cdot\frac{e^{2\sqrt{nz}}}{z^{\frac{1}{4}-\frac{a}{2}}n^{\frac{1}{4}+\frac{a}{2}}}\right)^{-1}\right)^{\sqrt{n}} = \\ = \frac{1}{48\sqrt{z}}\Big(3-12a^2+24(1-a)z+4z^2\Big)$$

for $z \in \mathbb{C}\setminus(-\infty,0]$, $a \in \mathbb{C}$, where $\mathrm{Re}(\sqrt{nz})$ denotes $\sqrt{n|z|}\cos\left(\frac{\theta}{2}\right)$ and $\theta := arg(z) \in (-\pi,\pi]$.

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## Omówienie

Autorzy prezentowanego artykułu, zainspirowani granicą (1) zamieszczoną w notce [College Math. J. **42** (2011), 328], przedstawiają tu wiele różnych nowych granic. Niektóre z nich to uogólnienia wspomnianej granicy (1), a inne stanowią jedynie próbę ujęcia takich uogólnień. Obszar badań i dyskusji rozszerzono na funkcje specjalne, między innymi funkcję gamma, oraz uogólnione wielomiany Laguerre'a.