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BROUWER'S THEOREM FOR A SQUARE ON THE BASIS OF HEX THEOREM

Abstract. This work is a continuation of author's work [1] on fixed points. In this work, Brouwer's theorem is proved on the basis of the Hex theorem. In the proof, the author uses, among other things, the lemma about no draw. Two proofs of this lemma are derived. The second proof is a modification of D. Gale's proof [2] and is based on the concept of *a walk* on the Hex board.

1. Introduction

The game of Hex, although it turned out to be one of the simplest, is also one of the most interesting board games in mathematical considerations. It is carried out on a diamond-shaped board composed of hexagonal fields. The dimensions of the board are usually 11 by 11 fields. Each of the two players has tokens of different colors. Players alternately place their tokens on the free spaces of the board so that the adjacent tokens form an uninterrupted sequence connecting the opposite sides of the board of their own color. The winner is the player who creates such a sequence first. The game of Hex was invented by the Danish poet

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and mathematician Piet Hein in 1942 while considering the problem of four colors (this problem still has no solution, except for the computer-assisted proof). Independently from Hein, the game of Hex was invented in 1948 by the American mathematician John Nash. The game gained popularity among students of American universities. Nash proved in 1949 that the game cannot end in a draw and, regardless of the board size, there is always a winning strategy for the player making the first move. Let us add that the lack of tie is due to the fact that one player can block the other player by completing his own sequence.

In this work, we deal with graphs, among other things. It is therefore worth to say a few words about their educational role. Already in this work, it can be noticed that by examining an appropriate graph, it is possible to decide whether there is a winning strategy in certain games (e.g. in the Hex game). It is worth to mention the following educational and engineering advantages of graphs:

- learning about the types of reasoning (including combinatorial and spatial reasoning),
- usefulness in the description of the world around us and its phenomena,
- assistance in solving practical and theoretical problems.

Let us add that some of the proofs presented in this work constitute a good illustration of the connections between different branches of mathematics, e.g. combinatorics and topology.

2. The game of Hex and Brouwer's fixed point theorem

Before we present the proof of the Hex theorem, we first clarify the rules of the game. Two players Alfa and Beta alternately place black and white tokens on empty cells of the board (Fig. 1). In each move Alfa places one black token and Beta places one white token. We assume that Alfa starts. The winner is the player who first creates a chain of his tokens connecting the corresponding two opposite sides of the board – the black sides for the player Alfa and the white sides for the player Beta. If all hexes are occupied by tokens and nobody created such a chain, then it is a draw.

An appealing feature of this game is that it never ends in a draw. Namely, there is a strategy in which Alfa tries to block Beta just by completing his own

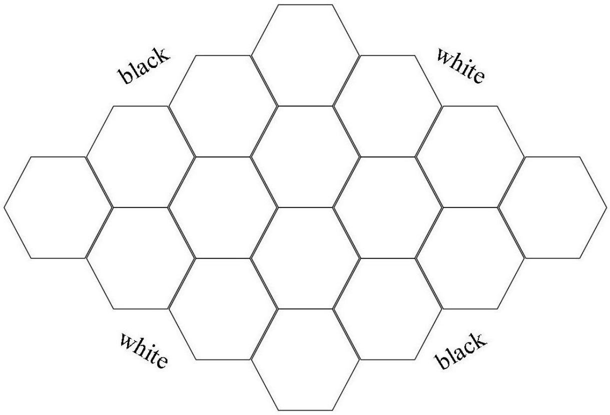

Fig. 1. Game board with dimension 4

chain. This can be compared to the situation where Alfa is trying to build a dam by stacking black stones corresponding to the black tokens, while Beta behaves like water breaking the dam. Thus the following question arises: can Alfa block the water effectively? We show that the answer is positive.

The method presented in the below proof of Hex theorem is called a *thief strategy* and is derived from J. Nash.

Theorem 1. *(Hex theorem) Let* Γ *be the game of Hex with* $n \times n$ *cells. Then the first player (Alfa) has a winning strategy.*

Proof. Observe that exactly one of the following three cases holds: Alfa has a winning strategy, Beta has a winning strategy or they both have a draw strategy. Suppose that Beta has a winning strategy. Then Alfa can start the game by tagging any cell (it cannot harm it) and then follow the Beta strategy with one extra cell occupied. Thus Alfa could win, which contradicts our assumption. Therefore, the second case is impossible, and it remains to prove that a draw is impossible. With this aim, it is enough to prove the next lemma. □

Lemma 2. *(about no draw) The Hex game cannot end in a draw.*

Below, we present two proofs of Lemma 2. The first proof is a modification of the proof from [3] and uses the so-called Sperner's lemma.

Lemma 3. *(E. Sperner, 1928) Let the triangle $A_1A_2A_3$ be divided into smaller triangles such that any pair of edges either coincides or has at most one common point. Assume that the vertices of the triangles from the division are numbered* $1, 2, 3$ *according to the Sperner's labeling, which means that the following condition holds:*

- $(*)$ *the vertex A_i $(i = 1, 2, 3)$ has number i and every vertex lying on the side A_iA_j $(1 \leq i, j \leq 3)$ has number i or j. The numbering of vertices inside the triangle $A_1A_2A_3$ is arbitrary.*

Then there is at least one triangle different from $A_1A_2A_3$ whose vertices are numbered $1, 2, 3$ *(Fig. 2).*

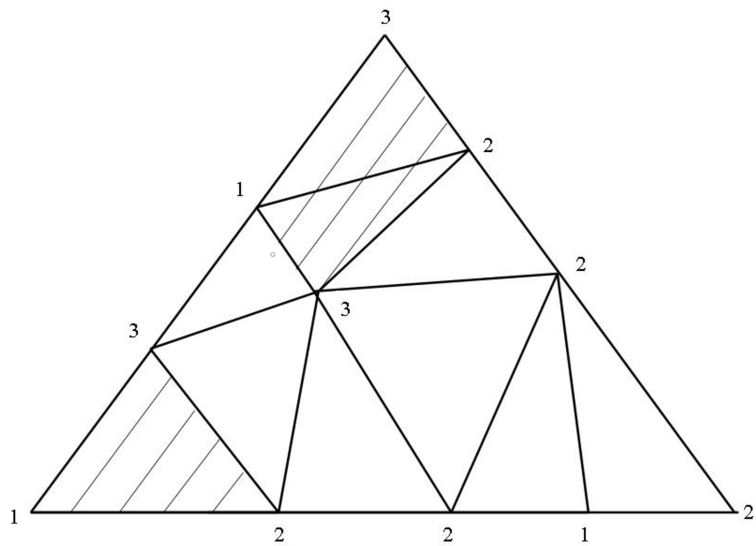

Fig. 2

First proof of Lemma 2. Assume contrary that the game ends in a draw. Then all hexes on the board are occupied by black and white tokens. We say that the cells with black tokens belong to Alfa, and the cells with white tokens belong to Beta. Every such a board naturally defines a graph whose vertices are the cells of the board, together with four artificial vertices w_1, w_2, w_3 and w_4, representing the sides of the board, where the vertices w_1 and w_3 represent the two opposite black sides (we say that w_1 and w_3 belong to Alfa), and the vertices w_2 and w_4 represent the two opposite white sides (we say that w_2 and w_4 belong to Beta). The graph corresponding to the exemplary 4×4 board with nine black tokens and seven white tokens is depicted in Fig. 3. The assumption that the game ends

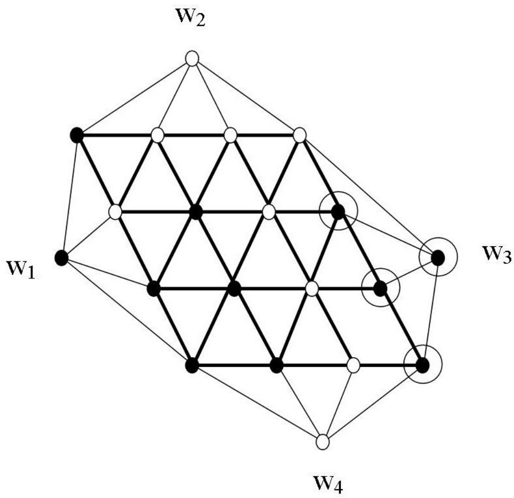

Fig. 3. Graph of the game with dimension 4

in a draw can be equivalently formulated as follows: there is no path belonging entirely to Alfa which connects w_1 and w_3, and there is no path belonging entirely to Beta which connects w_2 and w_4.

Let us number every vertex w with 0, 1, 2, according to the following rule:

- w has number 1 if it belongs to Alfa and there is a path connecting w_1 and w which belongs entirely to Alfa,
- w has number 0 if it belongs to Beta and there is a path connecting w_2 and w which belongs entirely to Beta,
- in all other cases w has number 2 (in Fig. 3 this is symbolized with a circle in a circle).

By deforming our graph, we may obtain the triangle Δ with the vertices w_1, w_2 and w_3 (the triangulation of Δ defined by the graph from Fig. 3 is presented in Fig. 4). It is easy to see that the vertices w_1, w_2, w_3 have numbers 1, 0, 2,

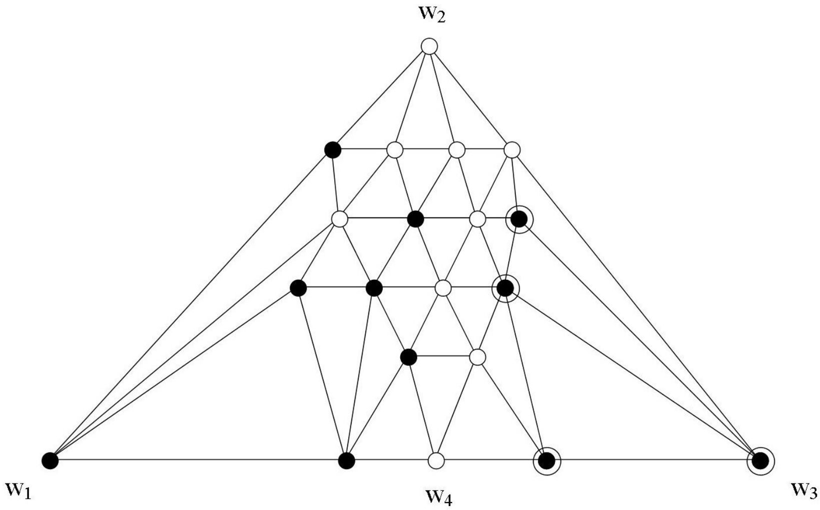

Fig. 4

respectively. We now show that the triangulation of Δ meets the assumptions of the Sperner's lemma.

The left side w_1w_2 of Δ contains three vertices: w_1, w_2 and the upper left corner of the board. This corner is marked black or white, depending on whether it belongs to Alfa or to Beta. In the first case, it has number 1, and in the second case, it has number 0. Similarly, the right side w_2w_3 of Δ contains three vertices: w_2, w_3 and the upper right corner of the board. If this corner belongs to Beta, then its number obviously cannot be 1, and if it belongs to Alfa, then its number also cannot be 1, because otherwise there would be a path belonging entirely to Alfa and connecting this corner with w_1. In consequence, there would be a path belonging entirely to Alfa and connecting w_1 and w_3, which contradicts our assumption. Finally, the side w_1w_3 of Δ contains five vertices: w_1, w_3, w_4, and the two bottom corners of the board. If w is any of these five vertices, then w can not be numbered with 0. This is obviously true in the case $w \in \{w_1, w_3\}$. If $w = w_4$ or if w is one of the corners, then it is also true, because otherwise, the vertex w would belong to Beta and there would be a path belonging entirely to Beta and connecting w with w_2. Since w_4 belongs to Beta and the bottom corners are connected with w_4, this would imply that there is a path belonging to Beta and connecting w_2 and w_4, contradiction.

Therefore, on the basis of the Sperner's lemma, there is a small triangle u_1, u_2, u_3 with the vertices numbered 1, 0, 2, respectively. Now, if u_3 belongs to Alfa, then, since there is a path from w_1 to u_1 belonging to Alfa, there is also a path from w_1 to u_3 belonging to Alfa, which implies that u_3 has number 1 and we have a contradiction. Analogously, we get a contradiction when u_3 belongs to Beta, which finishes this proof.

The second proof of Lemma 2 uses *walks* on the extended Hex board and constitutes a modification of D. Gale's proof from [2].

Second proof of Lemma 2. Assume that all cells on an $n \times n$ Hex board are occupied by black and white tokens. Let us stick to our board two new rows of cells, each having n hexagonal cells: one row along the left side and one row along the bottom side. To obtain an $(n+1) \times (n+1)$ Hex board (further called the extended board), we also need to fill the gap between the above two new rows of cells by one more corner cell (left-bottom corner cell). We place black tokens on the cells sticking to the left side, as well as on the new corner cell, and on the remaining n new cells, we place white tokens. The exemplary original 4×4 board and the corresponding extended 5×5 board are depicted in Fig. 5.

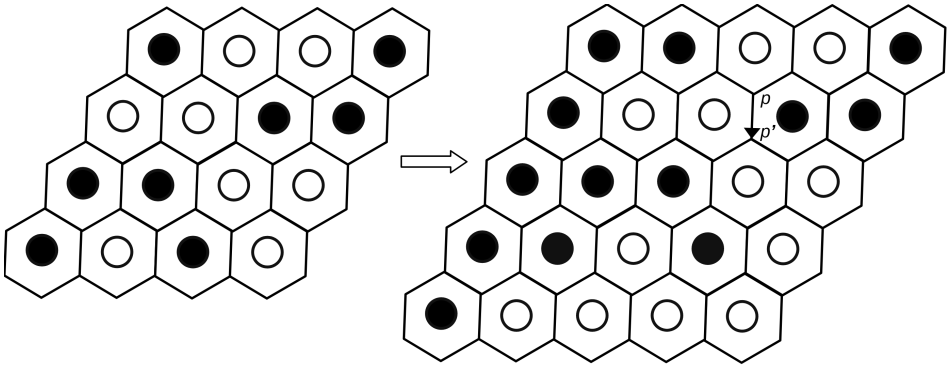

Fig. 5. The original 4×4 board and the extended 5×5 board

For the sides of cells on the extended board, we consider the following operation: if a side is the common side of two differently occupied adjacent cells, then we define its orientation so that the cell occupied by the black token is on the left. We get every such a side as a vector $\overrightarrow{pp'}$ with the beginning p and the end p' (an exemplary vector $\overrightarrow{pp'}$ is depicted in Fig. 5). The set of all such vectors satisfies the following properties:

(1) Any two different vectors cannot have the same end.

(2) There is a unique vector whose beginning belongs to the bottom side of the extended board. We denote it further by $\overrightarrow{p_0p_1}$ (see Fig. 6). This vector

belongs to the left-bottom corner cell and its end p_1 is an internal point, that is p_1 belongs to three pairwise adjacent cells.

(3) The end of any vector either is an internal point, or it belongs to the upper or to the right side of the board. Moreover, if the end of some vector is an internal point, then it is the beginning of exactly one other vector.

(4) Let $v_0, \ldots, v_s$ $(s \geq 0)$ be any sequence of vectors such that $v_0 := \overrightarrow{p_0 p_1}$ and the end of v_i is the beginning of v_{i+1}, that is, we can write $v_i = \overrightarrow{p_i p_{i+1}}$ for $i = 0, \ldots, s-1$. Then $v_i \neq v_j$ for $i \neq j$.

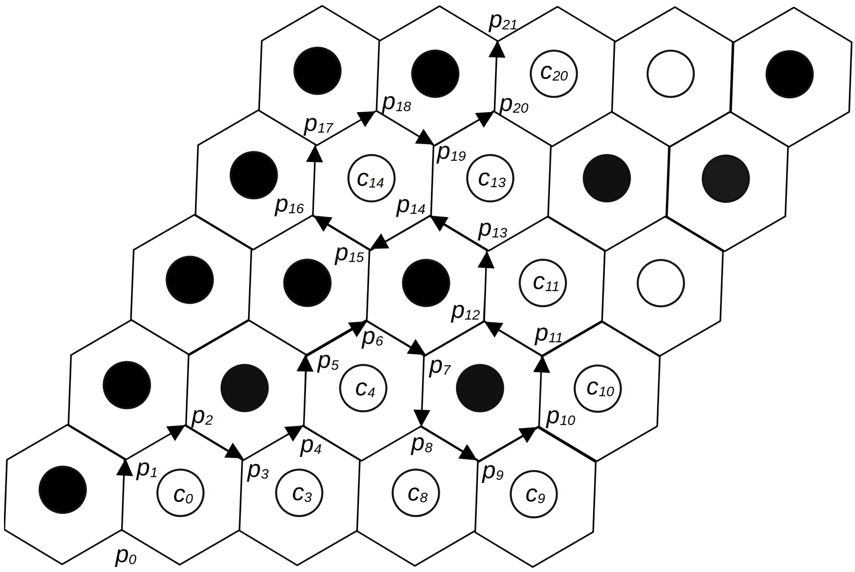

Fig. 6. The sequence of vectors $\overrightarrow{p_i p_{i+1}}$ defining a winning chain for Beta

Observations (1)–(3) are trivial. To show (4), suppose contrary that $v_k = v_l$ for some $0 \leq k < l \leq s$, where the value of k is possibly small. If $k = 0$, then $v_0 = v_l$ and the vertex $p_0 = p_l$ is the end of the vector v_{l-1}. But the vertex p_0 belongs to the bottom side of the board and, by observation (3), it cannot be the end of any vector. Thus, it must be $k > 0$ and, since $v_k = v_l$, the vectors v_{k-1}, v_{l-1} are different and have the same end (which is the vertex p_k), but this is impossible by observation (1).

Since the set of all vectors is finite, we obtain by observations (3)–(4) that there is a unique sequence $v_0, \ldots, v_{s_0}$ of vectors $v_i = \overrightarrow{p_i p_{i+1}}$ $(i = 0, 1, \ldots, s_0 - 1)$

such that the end p_{s_0+1} of v_{s_0} belongs to the upper side or to the right side of the board (see Fig. 6). The sequence $v_0, \ldots, v_{s_0}$ may be interpreted as a walk along the sides of the cells, which starts at the left-bottom corner cell and finishes at the upper or at the right side of the board. This walk defines the winning chain for Alpha or for Beta, depending on whether the end of v_{s_0} belongs to the right or to the upper side of the board, respectively. For example, if the end of v_{s_0} belongs to the upper side (as in Fig. 6), we get the sequence $c_0, \ldots, c_{s_0}$, where c_i is the unique cell with a white token directly on the right side of the vector $v_i = \overrightarrow{p_i p_{i+1}}$. The sequence $c_0, \ldots, c_{s_0}$ contains repetitions, but two different consecutive elements of this sequence must be adjacent cells, and hence, there is the shortest subsequence $c_{t_0}, \ldots, c_{t_l}$ satisfying: $t_0 = 0$, $t_l = s_0$ and for every $0 \leq j < l$ the cells c_{t_j} and $c_{t_{j+1}}$ are adjacent. Then the winning chain for Beta on the original $n \times n$ board is $c_{t_{j_0+1}}, c_{t_{j_0+2}}, \ldots, c_{t_l}$, where $j_0 \in \{0, 1, \ldots, l-1\}$ is the greatest index such that the cell $c_{t_{j_0}}$ touches the bottom side of the extended board. For example, for the extended board from Fig. 5, the sequence $v_0, v_1, \ldots, v_{20}$ of vectors $v_i = \overrightarrow{p_i p_{i+1}}$ is depicted in Fig. 6, the corresponding shortest subsequence of the sequence $c_0, \ldots, c_{20}$ is $c_0, c_3, c_8, c_9, c_{10}, c_{11}, c_{13}, c_{20}$, and the winning chain for Beta on the original 4×4 board is $c_{10}, c_{11}, c_{13}, c_{20}$.

Below, we present the proof of the Brouwer's fixed point theorem, which uses the lemma about no draw. The similar proof is presented in [2].

Theorem 4. (Brouwer's fixed point theorem) *Every continuous mapping from a unit square $I^2 = \langle 0, 1\rangle \times \langle 0, 1\rangle$ into itself has a fixed point (i.e. there is a point $x \in I^2$ such that $f(x) = x$).*

Proof. For every natural number n, we consider the subset

$$V_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\} \times \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\} \subseteq I^2$$

as the vertex set of the graph Γ_n in which two vertices $x, y \in V_n$ are connected by an edge if and only if their distance $d(x, y)$ belongs to the set $\{1/n, \sqrt{2}/n\}$ and if $d(x, y) = \sqrt{2}/n$, then the line connecting x and y has positive slope. The graph Γ_n may serve as the board for the Hex game (so-called dual board – see Fig. 7). The left (right) side of this board form the vertices $(0, i/n)$ (respectively: $(1, i/n)$) for $i = 0, 1, \ldots, n$. Similarly, the bottom (upper) side of the board form the vertices $(i/n, 0)$ (respectively: $(i/n, 1)$) for $i = 0, 1, \ldots, n$. The players place tokens on the vertices of Γ_n and one of them tries to build a path connecting the left side with the right side, and the other wants to build a path connecting the

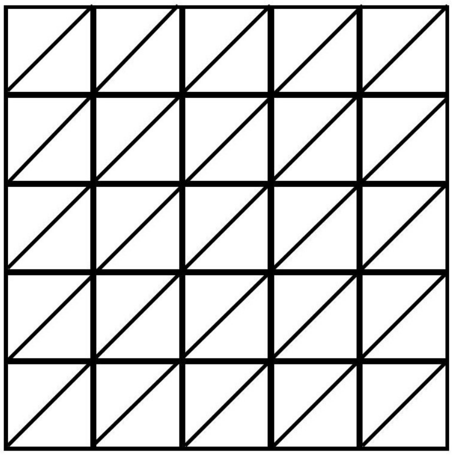

Fig. 7. Dual board Γ_5

bottom side with the upper side. Now, by the lemma about no draw, we see that for any two subsets $A, B \subseteq V_n$ the equality $V_n = A \cup B$ implies that A contains a path connecting the left and right sides or B contains a path connecting the bottom and upper sides.

Since I^2 is a compact set, from any infinite sequence (x_n) of points $x_n \in I^2$ one can choose a convergent subsequence (x_{n_i}). If we denote $x^* = \lim_{i\to\infty} x_{n_i}$, then by the triangle inequality, we get:

$$d\left(f(x^*), x^*\right) \leqslant d\left(f(x^*), f(x_{n_i})\right) + d\left(f(x_{n_i}), x_{n_i}\right) + d(x_{n_i}, x^*).$$

We obviously have $\lim_{i\to\infty} d(x_{n_i}, x^*) = 0$. Since f is continuous, we also have $\lim_{i\to\infty} d(f(x^*), f(x_{n_i})) = 0$. Thus, if $\lim_{i\to\infty} d\left(f(x_{n_i}), x_{n_i}\right) = 0$, then we get $d\left(f(x^*), x^*\right) = 0$. In consequence, it remains to show that there exists a sequence (x_n) of points from I^2 such that $\lim_{n\to\infty} d\left(f(x_n), x_n\right) = 0$. Then the required fixed point of f is the limit of any convergent subsequence of (x_n).

Let $\varepsilon > 0$ be any positive number. By the above, it is enough to prove that there is $x_\varepsilon \in I^2$ such that $d\left(f(x_\varepsilon), x_\varepsilon\right) < \varepsilon$. Since the function f is continuous and I^2 is compact, f is also uniformly continuous, so there exists $0 < \delta < \varepsilon$ such that for all $x, y \in I^2$ the inequality $d(x, y) < \delta$ implies $d(f(x), f(y)) < \varepsilon$.

For every $x = (x_1, x_2) \in I^2$, we denote

$$x' = (x_1', x_2') = f(x) \in I^2.$$

Fix a natural number n such that

$$\frac{1}{n} < \delta(\sqrt{2} - 1) < \frac{\delta}{\sqrt{2}}.$$

Let $P^{\rightarrow} \subseteq V_n$ be the subset (perhaps empty) of vertices $x = (x_1, x_2)$ that f moves to the right by at least $\varepsilon_0 := \varepsilon/\sqrt{2}$, i.e.

$$P^{\rightarrow} = \{(x_1, x_2) \in V_n \colon x_1' - x_1 \geqslant \varepsilon_0\}.$$

Similarly, we define the subsets $P^{\leftarrow}, P^{\downarrow}, P^{\uparrow} \subseteq V_n$. Note that $P^{\leftarrow} \cap P^{\rightarrow} = \emptyset$ and $P^{\downarrow} \cap P^{\uparrow} = \emptyset$. Let us fix two vertices $x = (x_1, x_2) \in P^{\rightarrow}$ and $y = (y_1, y_2) \in P^{\leftarrow}$. We have

$$x_1' - x_1 \geqslant \varepsilon_0, \quad y_1 - y_1' \geqslant \varepsilon_0.$$

Suppose that x and y are connected by an edge in the graph Γ_n. Then their horizontal coordinates x_1 and y_1 differ by at most $\frac{1}{n}$. Therefore

$$x_1 - y_1 \geqslant -\frac{1}{n}.$$

Adding the above three inequalities, we get

$$x_1' - y_1' \geqslant 2\varepsilon_0 - \frac{1}{n} > 2\varepsilon_0 - \delta(\sqrt{2} - 1) > 2\varepsilon_0 - \varepsilon(\sqrt{2} - 1) = \varepsilon.$$

In consequence $d(x', y') \geq x_1' - y_1' > \varepsilon$. On the other hand, since x and y are connected by an edge, we get $d(x, y) \leq \sqrt{2}/n < \delta$, and, since f is uniformly continuous, we get $d(x', y') < \varepsilon$, contradiction.

Therefore, if $x \in P^{\rightarrow}$ and $y \in P^{\leftarrow}$, then x and y cannot be adjacent. Moreover, if $x = (x_1, x_2) \in P^{\rightarrow}$, then $x_1 \leqslant x_1' - \varepsilon_0 < 1$. Hence, the set $P^{\rightarrow}$ does not contain vertices from the right side of the board. Similarly, the set $P^{\leftarrow}$ does not contain the vertices from the left side of the board. Therefore, the set $A := P^{\rightarrow} \cup P^{\leftarrow}$ cannot contain the winning path for the player wishing to join the left and right sides of the board. Similarly, the set $B := P^{\uparrow} \cup P^{\downarrow}$ cannot contain the winning path for the other player. In consequence, by the lemma about no draw, we get $A \cup B \neq V_n$, which means that the set $S := V_n \setminus (P^{\uparrow} \cup P^{\downarrow} \cup P^{\rightarrow} \cup P^{\leftarrow})$ is not empty. Let $x_\varepsilon = (x_1, x_2) \in S$. From the definition of the sets $P^{\uparrow}, P^{\downarrow}, P^{\rightarrow}, P^{\leftarrow}$, it follows that the image $f(x_\varepsilon) = (x'_1, x'_2) \in I^2$ satisfies:

$$-\varepsilon_0 < x'_1 - x_1 < \varepsilon_0,$$
$$-\varepsilon_0 < x'_2 - x_2 < \varepsilon_0.$$

Thus $f(x_\varepsilon)$ is contained inside the square with the center x_ε and side length $2\varepsilon_0$. Therefore $d(f(x_\varepsilon), x_\varepsilon) < \varepsilon_0\sqrt{2} = \varepsilon$. □

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