

Zygmunt Zahorski and contemporary real analysis

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Professor Zygmunt Zahorski was an eminent specialist in real analysis. His papers, concerning mainly different classes of real functions of a real variable, are precise and sophisticated (in the positive meaning of the word). Numerous mathematicians throughout the world are working on real functions theory using his ideas and results. Among most widely known are: his first pupil professor Jan Stanisław Lipiński and American mathematician Andrew M. Bruckner from Santa Barbara, California. Since the number of papers inspired by the results, ideas or techniques of professor Zahorski exceeds hundreds, in the sequel we shall concentrate on some chosen publications. An essential part of mathematical activity of professor Lipiński was strictly connected with the kind of problems considered by professor Zahorski. His achievements deserve a separate presentation. Fortunately, quite recently Paul Humke has done an excellent work (see [30]).

The most frequently quoted and the most influential paper of professor Zygmunt Zahorski is the monumental treatise on the first derivative ([6]). A lot of very interesting facts on Dini derivatives is included in the habilitation thesis, which, however, was never published.

In the paper [6] professor Zahorski tried to characterize sets of the form $\{x: f(x) < a\}$ and $\{x: f(x) > a\}$, when f is a real function of a real variable possessing a continuous primitive function. To this end he considered two hierarchies: of sets $M_0 \supset M_1 \supset M_2 \supset M_3 \supset M_4 \supset M_5$ and of functions $\mathcal{M}_0 \supset \mathcal{M}_1 \supset \mathcal{M}_2 \supset \mathcal{M}_3 \supset \mathcal{M}_4 \supset \mathcal{M}_5$.

Let $E \subset \mathbb{R}$ be a non-empty F_σ set. We say that E belongs to the class

- M_0 if each point of E is a point of bilateral accumulation of E ,
- M_1 if each point of E is a point of bilateral condensation of E ,
- M_2 if each one sided neighbourhood of each $x \in E$ intersects E in a set of positive measure,
- M_3 if there exists a sequence $\{K_n\}_{n \in \mathbb{N}}$ of closed sets and a sequence $\{\eta_n\}_{n \in \mathbb{N}}$ of numbers from $[0, 1)$ such that $E = \bigcup_n K_n$ and for each $x \in K_n$ and each $c > 0$ there exists a number $\epsilon(x, c) > 0$ such that if h and h_1 satisfy $hh_1 > 0$, $\frac{h}{h_1} < c$, $|h + h_1| < \epsilon(x, c)$, then

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$$\frac{\lambda(E \cap (x + h, x + h + h_1))}{h_1} > \eta_n,$$

- M_4 if E fulfills M_3 and each η_n is positive,
- M_5 if each point of E is a point of density of E .

Let f be an extended real valued function defined on some interval I . We say that $f \in \mathcal{M}_k$ if $\{x : f(x) < a\} \in M_k$ and $\{x : f(x) > a\} \in M_k$ for each $a \in \mathbb{R}$, $k = 0, 1, \dots, 5$. The above sets usually are called the associated sets of f . Professor Zahorski has proved that all inclusions between classes of sets and of functions are strict with one exception: $\mathcal{M}_0 = \mathcal{M}_1$ and this is a class of Darboux Baire one functions. Also in [6] it is proved that \mathcal{M}_5 is the class of approximately continuous functions. It is interesting that approximately continuous functions were considered already by A. Denjoy around 1915 and the density topology was introduced about 40 years later by O. Haupt and Ch. Pauc ([29]). Recall that the real function of a real variable is approximately continuous if it is continuous when the ordinary topology is used on the range and the density topology is used on the domain. One can find more information in [55]. Main results of the paper [6] are connected with the conditions M_2 , M_3 and M_4 . Namely, professor Zahorski has proved that if f is a derivative (possibly infinite) of a continuous function, then all associated sets of f belong to M_2 , if f is a finite derivative, then all associated sets belong to M_3 and if f is a bounded derivative, then all associated sets are in M_4 . None of these necessary conditions is sufficient. More detailed information can be found in [14] and [17], where is also an exhaustive list of references.

Now it is time to explain why professor Zahorski in [6] has studied the characterization of sets associated with the derivative and not the characterization of the derivative in terms of associated sets. Numerous classes of functions are characterized in that way, for example a function is continuous if and only if the associated sets are open, a function is Baire one if and only if they are F_σ , a function is measurable if and only if they are measurable and so on. Observe that all mentioned classes are closed with respect to superpositions from outside with a homeomorphism, i.e. if f is in the class and $h : \mathbb{R} \xrightarrow{\text{onto}} \mathbb{R}$ is a homeomorphism, then $h \circ f$ is in the same class. The class of derivatives (bounded or unbounded, finite or assuming infinite values) is far from possessing this property. This was observed by G. Choquet in [21] who proved that if h is a nonlinear homeomorphism $h : \mathbb{R} \xrightarrow{\text{onto}} \mathbb{R}$ then there exists a bounded derivative f_0 such that $h \circ f_0$ is not a derivative. Professor Zahorski was obviously aware of this fact. A. Bruckner in [15] deeply studied the bad behaviour of the derivative with this respect and proved that there exists a bounded derivative f such that for each nowhere linear (linear on no interval) homeomorphism $h : \mathbb{R} \xrightarrow{\text{onto}} \mathbb{R}$ and each interval $I \subset \mathbb{R}$ there exists a point $x \in I$ such that $h \circ f$ is not the derivative of its integral at x .

The decisive step in characterizing the associated sets of the derivative has been made by D. Preiss in [42]. His paper is devoted to the characterization of the triple S, G, E of subsets of the real line for which there exists a function $f : \mathbb{R} \rightarrow \mathbb{R}$ differentiable at each point such that $E = \{x : f'(x) > 0\}$, $G = \{x : f'(x) = +\infty\}$ and S is the set of points of discontinuity of f . Preiss has defined the class of sets M^* (the definition is rather complicated and uses the condition similar to M_4 of Zahorski) and

has proved that M^* is the class of associated sets for not necessarily finite derivatives whose primitives need not be continuous and $M_2^* = M^* \cap M_2$ is the class of associated sets for not necessarily finite derivatives whose primitives are continuous and $M_3^* = M^* \cap M_3$ is the class of associate sets for finite derivatives. His results have also application to the approximative derivatives. C. Neugebauer in [37] has proved the characterization of derivatives in terms of the behaviour of interval functions. His theorems say that a function $f : I_0 \rightarrow \mathbb{R}$ (where I_0 is some interval) is Darboux Baire one if and only if it fulfills the following condition C_1 : for each interval $I \subset I_0$ there exists a point $x_I \in \text{Int } I$ such that $I \rightarrow x$ implies $f(x_I) \rightarrow f(x)$ (here $I \rightarrow x$ means that $x \in I$ and $\lambda(I) \rightarrow 0$) and a function $f : I_0 \rightarrow \mathbb{R}$ is the derivative if and only if it fulfills the condition C_1 and moreover if $I = I_1 \cup I_2$, $\text{Int } I_1 \cap \text{Int } I_2 = \emptyset$, then (for x_I from C_1)

$$f(x_I) = \frac{f(x_{I_1}) \cdot \lambda(I_1) + f(x_{I_2}) \cdot \lambda(I_2)}{\lambda(I)}.$$

The last equality means that $f(x_I) \cdot \lambda(I)$ is an additive interval function. The theorem of Neugebauer shows how much Darboux Baire one functions differ from the derivatives.

D. Preiss and M. Tartaglia in [43] have given an interesting characterization of derivatives in terms of the set of derivatives (a sort of circular characterization according to Ch. Freiling ([27])). They proved that f is a derivative if and only if for each set $E \subset \mathbb{R}$ there is a derivative g such that $f^{-1}(E) = g^{-1}(E)$. Continuing this way K. Ciesielski in [22] has proved a general theorem stating that numerous families \mathcal{F} of real functions (including the family Δ of all derivatives) can be characterized as a family of the form $C(\mathcal{D}, \mathcal{A}) = \{f \in \mathbb{R}^{\mathbb{R}} : f^{-1}(A) \in \mathcal{D} \text{ for every } A \in \mathcal{A}\}$, where \mathcal{A} is some family of subsets of \mathbb{R} and $\mathcal{D} = \{f^{-1}(A) : f \in \mathcal{F} \text{ and } A \in \mathcal{A}\}$. His main theorem reads as follows: Let \mathcal{F}, \mathcal{R} be such that $\text{card}(\mathcal{R}) \leq \mathfrak{C}^+$, $\text{card}(\mathcal{F}) \leq \mathfrak{C}$, where \mathfrak{C} denotes the cardinality of continuum and \mathfrak{C}^+ is the next cardinal number, \mathcal{F} contains all constant functions and $\text{card}(g(\mathcal{R})) = \mathfrak{C}$ for any non-constant function g which is a difference of two functions from \mathcal{F} . Then there exists a family $\mathcal{A} \subset 2^{\mathbb{R}}$ of cardinality less or equal to $\text{card}(\mathcal{R})$ such that $\mathcal{F} \cap \mathcal{R} = \mathcal{R} \cap C(\mathcal{D}, \mathcal{A})$, where $\mathcal{D} = \{f^{-1}(A) : f \in \mathcal{F} \text{ and } A \in \mathcal{A}\}$ as before. If we take $\mathcal{F} = \Delta$ and \mathcal{R} – the family of Borel functions, we conclude that there exists a family $\mathcal{A} \subset 2^{\mathbb{R}}$ such that $\text{card}(\mathcal{A}) \leq \mathfrak{C}$ and $\Delta = \mathcal{R} \cap C(\mathcal{D}, \mathcal{A})$, where $\mathcal{D} = \{f^{-1}(A) : f \in \Delta \text{ and } A \in \mathcal{R}\}$. Ciesielski also proved in [22] that there exists a Bernstein set $B \subset \mathbb{R}$ such that $\Delta = \mathcal{D}\mathcal{B}_1 \cap C(\mathcal{D}_0, \{B+c : c \in \mathbb{R}\}) = C(\mathcal{D}, \mathcal{A})$, where $\mathcal{A} = \bigcup_{c \in \mathbb{R}} \{(-\infty, c), (c, \infty), B+c\}$, $\mathcal{D}_0 = \{f^{-1}(B+c) : f \in \Delta \text{ and } c \in \mathbb{R}\}$ and $\mathcal{D} = \{f^{-1}(A) : f \in \Delta \text{ and } A \in \mathcal{A}\}$, so the family consisting of all translations of a single Bernstein set is sufficient.

Recall the definition of the Kurzweil-Henstock integral. Let I be a closed interval, I_1, \dots, I_n – a partition of I and x_1, \dots, x_n – a sequence of points such that x_i belongs to the interval I_i for each i . Such system of intervals and points is called a tagged partition of I . Suppose that f is any function defined on I then each tagged partition yields a Riemann sum given by $\sum_{i=1}^n f(x_i) \cdot \lambda(I_i)$. If δ is a positive function defined on I and for each $i \in \{1, \dots, n\}$ we have $\lambda(I_i) < \delta(x_i)$, then the tagged partition is called δ -fine (such positive function is usually called gauge function). A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is Kurzweil-Henstock integrable if and only if for each closed interval I and for each $\epsilon > 0$ there exists a gauge $\delta : I \rightarrow \mathbb{R}^+$ such that any two δ -fine tagged partitions of I have Riemann sums which differ by less than $\epsilon \cdot \lambda(I)$. The LH-integral is then

defined to be the limit of the corresponding Riemann sums as $\epsilon \rightarrow 0$. It is known that derivatives are KH-integrable. Ch. Freiling in [27] has observed that in fact a function f is a derivative if and only if it is KH-intergrable. The paper [27] is a good source of informations about possible characterizations of derivatives.

Let us come back for a moment to the class M_3 . C. Weil in [53] has introduced the property Z and has proved that if a function has the Darboux and Denjoy property, then the property Z implies the Zahorski property M_3 . Moreover, derivatives, approximate derivatives, L_p -derivatives all have the property Z . P.S. Bullen and D.N. Sarkhel have made a step further – they have defined the property Z^* (stronger than Z) in the following way:

The function f on I is said to have the property Z^* if for every $c \in I$ and $\epsilon > 0$, $\eta > 0$ there is a neighbourhood I_c of c such that the following conditions Z^+ and Z^- hold:

$$Z^+: \text{ if } f(x) \geq f(c) - \epsilon \quad \text{a.e. on a closed interval } J \subset I_c, \text{ then } \lambda(A) - \lambda(B) \leq \eta \cdot \rho(c, J) \quad (\rho(c, J) - \text{a distance between } c \text{ and } J),$$

where $A = \{x \in J : f(x) \geq f(c) + \epsilon\}$, $B = \{x \in J : f(c) - \epsilon \leq f(x) < f(c)\}$,

$$Z^-: \text{ if } f(x) \leq f(c) + \epsilon \quad \text{a.e. on a closed interval } J \subset I_c, \text{ then } \lambda(A) - \lambda(B) \leq \eta \cdot \rho(c, J),$$

where $A = \{x \in J : f(x) \leq f(c) - \epsilon\}$, $B = \{x \in J : f(c) < f(x) \leq f(c) + \epsilon\}$.

The main result of [19] says that k -th Peano derivative, k -th approximate Peano derivative and k -th L_p -derivative all have the property Z^* (for $k \geq 1$).

Still in the paper [6] one can find the following theorem:

Theorem. *Let f be a function fulfilling on an interval I the following conditions:*

- (i) f is a Darboux function,
- (ii) f' exists (finite or not) possibly except a denumerable set of points,
- (iii) $f'(x) \geq 0$ a.e.

Then f is continuous and nondecreasing on I .

In 1939 G. Tolstov ([49]) has proved the theorem which is an improvement of the theorem of Goldowski-Tonelli:

Theorem. *Let f be a function fulfilling on an interval I the following condition:*

- i) f is approximately continuous,
- ii) f'_{ap} exists (finite or not) possibly except a denumerable set of points,
- iii) $f'_{ap}(x) \geq 0$ a.e.

Then f is continuous and nondecreasing on I .

Observe that the condition (i) in Zahorski's theorem is weaker than in Tolstov's while conditions (ii) and (iii) are stronger because they involve ordinary derivative instead of the approximate derivative. Professor Zahorski asked if it is possible to prove a theorem which implies both Tolstov's theorem and Zahorski's theorem. However, there exists a non-monotone function which is Darboux and fulfills condition (ii) and (iii) of Tolstov (see [17], p. 45), so simply taking the weaker condition from each pair does not work. From the second condition of Zahorski it follows that f is Baire one

function. T. Świątkowski in [48] and A. Bruckner in [11] and [12] have proved that if f is Darboux Baire one function on I and fulfills conditions (ii) and (iii) of Tolstov, then f is continuous and nondecreasing on I . In fact, A. Bruckner has proved the more general scheme:

Let \mathcal{P} be a function-theoretic property sufficiently strong to imply

- (a) any Darboux Baire one function which satisfies property \mathcal{P} on an interval I is of generalized bounded variation on I ,
- (b) any continuous function of bounded variation which satisfies property \mathcal{P} on I is nondecreasing on I .

Then any Darboux Baire one function which satisfies property \mathcal{P} on I is continuous and increasing on I .

Using this result E. Łazarow and W. Wilczyński have proved a similar theorem for the category analogue of the approximate derivative ([36]). For further informations on monotonicity conditions see [23]. E. Łazarow has proved in [35] that a finite \mathcal{I} -approximate derivative is Baire one.

R. Pawlak in [40] when studying Darboux and Świątkowski real valued functions of two real variables has introduced the hierarchy of Zahorski classes on arcs in \mathbb{R}^2 and has proved, among others, that the same inclusions and equalities between these classes, Darboux Baire one functions and approximately continuous functions of two variables hold as in the case of functions of one variable.

The problem of characterization of the set of points of nondifferentiability of a continuous functions has been solved by professor Zahorski in [2] and [3]. He has shown that this set is the union of a G_δ set with a $G_{\delta\sigma}$ set of Lebesgue measure zero and that any set of this form is the set of points of nondifferentiability for some continuous function. For a continuous function of bounded variation the term G_δ can be dropped from the statement. This theorem has been extended by A. Brudno ([18]) to arbitrary functions. The construction of a continuous function with prescribed set of points of differentiability has been simplified by S. Piranian ([41]). Observe that classical constructions of Bolzano, Weierstrass or van der Waerden are dealing with only one set of points of nondifferentiability of a continuous function, namely, the set equal to \mathbb{R} . The beautiful result of professor Zahorski for a long time was not commonly recognized. In the book *Real and Abstract Analysis*, Springer-Verlag 1969 by E. Hewitt and K. Stromberg one can find on page 266 the following sentences: “(17.13) Question. Suppose that $\lambda(A) = 0$, $A \subset [a, b]$. Is it possible to find a monotone function f on $[a, b]$ such that f' exists exactly on $A' \cap (a, b)$? The complete answer seems to be unknown”.

F.M. Filipczak in [24] has studied the set of points of differentiability from slightly another point of view. He proved that if E, F, G and H are subsets of an interval I such that $E \subset F \subset G \subset H \subset I$, E is of type F_σ , H is simultaneously of type F_σ and G_δ , $\lambda(H \setminus E) = 0$, $H \setminus F$ is countable (so F and G are G_σ 's), then there exists a real function defined on I such that E is the set where f' exists finite, F is the set of points of continuity of f and G is the set where f' exists finite or not. In [25] and [26] F.M. Filipczak has established the Borel class of symmetric derivatives of approximately continuous functions and has proved that the set of points of symmetric nondifferentiability is characterized exactly as in the case of ordinary nondifferentiability. His theorem reads as follows: If E is the set of the form

$E = A \cup B$, where $A \in G_\delta$, $B \in G_{\delta\sigma}$ and $\lambda(B) = 0$, then there exist a continuous function f such that f' exists and is finite for $x \notin E$, a symmetric derivative Df and unilateral derivatives do not exist for $x \in E$.

Professor Zygmunt Zahorski was interested also in the behaviour of more regular functions, namely, functions belonging to the class C^∞ . If $f \in C^\infty$ is a real function of a real variable and $Tf(x, h) := f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \dots$ associated Taylor series, then there are three possibilities: either the radius of convergence of T is positive and the series is convergent to f in some neighbourhood of 0, or the radius of convergence equals zero, or the radius of convergence is positive but the series does not converge to f . In the first case we say that x is a regular point (or a point of analyticity), in the second we say that x is a singular point in the sense of Pringsheim (or a point of divergence), in the third we say that x is a singular point in the sense of Cauchy (or a point of false convergence). The paper [4] contains an elegant and complete characterization of three sets. The theorem of Zahorski says that if $f \in C^\infty$, then the set A of regular points is open, the set D of points singular in the sense of Pringsheim is of type G_δ , the set F of points singular in the sense of Cauchy is the set of the first category of type F_σ and that if A, D, F are three disjoint sets such that $\mathbb{R} = A \cup D \cup F$, A is open, D is G_δ and F is F_σ of the first category, then there exists a function $f \in C^\infty$ for which A is the set of regular points, D – the set of points singular in the sense of Pringsheim and F – the set of points singular in the sense of Cauchy. The proof has been simplified by H. Salzmann and K. Zeller in [45]. J. Siciak in [46] using the method of these authors, has obtained an analogous result for functions of several variables.

Professor Zahorski has obtained also interesting results in differential geometry. In [5] he has proved among others that if K is a rectifiable curve in \mathbb{R}^2 , then there exists a parametric representation for K each of whose coordinate functions has a bounded derivative. Essentially the same result has been obtained by G. Choquet in [21]. More informations on this topic one can find in [13]. The paper [7] contains an unexpected construction of the very winding curve – the tangent line assumes all directions on each subarc of the curve. Moreover, the tangent line does not exist on dense set of points – it follows from the properties of the derivative. W. Wilczyński in [54] has presented a construction of a continuous function f defined on the unit circle $K = \{(x, y) : x^2 + y^2 \leq 1\}$ the graph of which is a rectifiable surface. The normal line to this surface takes every direction (from the upper semi-sphere) on each part of the surface. In this case also the set of non-differentiability of f must be dense of K .

Another important result concerning derivatives is contained in [1]. Professor Zahorski has constructed an everywhere differentiable continuous function with an infinite derivative on an arbitrary given G_δ set of Lebesgue measure zero. Earlier it was known that there exists a continuous function with an infinite derivative on an arbitrary G_δ null set and with finite Dini derivatives elsewhere. V. Tzodiks in [50] has proved the following related result: A necessary and sufficient conditions for two sets E_1 and E_2 to be sets where $f' = +\infty$ and $f' = -\infty$, where f is a finite function, are: E_1 and E_2 be $F_{\sigma\delta}$'s of measurable zero such that there exist disjoint F_σ sets H_1 and H_2 with $E_1 \subset H_1$ and $E_2 \subset H_2$.

T. Nishiura in [38] has used Zahorski classes of sets in the theory of absolute measurable spaces and absolute null spaces. A separable metrizable space is a Zahorski space if it is empty or it is the union of a countable sequence of topological copies

of the Cantor set. A subset of a separable metrizable space is a Zahorski set if it is a Zahorski subspace of this space. A Zahorski measure determined by the set E , where E is a Zahorski set in a separable metrizable space X is a continuous, complete, finite Borel measure μ on X such that $\mu(X \setminus E) = 0$ and $\mu(E \cap U) > 0$ if U is an open set such that $E \cap U \neq \emptyset$. The reader can observe immediately the analogy with the classes M_1 and M_2 . T. Nishiura has shown the relationship between Zahorski set and Lusin set and has expressed the opinion ([38], p. 193): “Zahorski spaces appear in a very prominent way in many proofs”.

According to the opinion of Professor Zygmunt Zahorski (see his biography in *Zeszyty Naukowe Politechniki Śląskiej Matematyka-Fizyka*, z. 48, Gliwice 1986, p. 19) among his publications there is only one “of essential good quality”. He did not say which one he had in mind. The international mathematical community duly appreciates numerous theorems of Zahorski, their influence in the development of the theory of real functions and orthogonal expansions, so it is really difficult to say what is his greatest achievement. I believe that it may be the construction of the rearrangement of terms of a Fourier series ([8]). In 1927 A. Kolmogorov in the paper [33] common with D. Menshov stated the following theorem: There exists a function $f \in L^2[0, 2\pi]$ whose Fourier series after some rearrangement of terms diverges almost everywhere. In spite of efforts of Kolmogorov himself and of his students the proof was still unattainable until 1960, when professor Zahorski accomplished the construction of the series and the rearrangement. Later P.L. Ulyanov ([51]) observed that similar construction works for the Walsh and the Haar system and A.M. Olevskii ([39]) and P.L. Ulyanov ([52]) proved that for any complete, orthogonal, normal system there exists a function $f \in L^2$ whose Fourier series with respect to this system after some rearrangement of terms diverges almost everywhere. Professor Zahorski set a high value on his rearrangement result. He used to mention a theorem of A.M. Garsia [28] which says that the existence of such permutation is highly improbable. Namely, let $f \in L^2[0, 2\pi]$ and let $\{m_k\}_{k \in \mathbb{N}}$ be an increasing sequence of positive integers such that $S_{m_k}(x, f) \xrightarrow[k \rightarrow \infty]{} f(x)$ almost everywhere. (Here $S_m(x, f)$ denotes the m -th partial sum of the Fourier series f). The existence of a sequence $\{m_k\}_{k \in \mathbb{N}}$ is assured by the following theorem (see [9], p. 178-181): if $\sum_{k=1}^{\infty} \frac{1}{m_k} < +\infty$ and $\sum_{k=n}^{\infty} \frac{1}{m_k} = O(\frac{1}{m_n})$ and $f \in L^2[0, 2\pi]$, then $S_{m_k}(x, f) \xrightarrow[k \rightarrow \infty]{} f(x)$ almost everywhere. Consider the permutation $\sigma = \{\sigma_1, \sigma_2, \dots\}$ of the natural numbers related to the sequence $\{m_k\}_{k \in \mathbb{N}}$ in the following way: if $m_{k-1} < i \leq m_k$, then $m_{k-1} < \sigma_i \leq m_k$ (we assume $m_0 = 0$). Let \mathcal{P}_k be the set of all permutations of $\{m_{k-1} + 1, m_{k-1} + 2, \dots, m_k\}$. The set \mathcal{P} of all permutations σ of the natural numbers described above can be naturally identified with the direct product of $\{\mathcal{P}_k\}_{k \in \mathbb{N}}$. If μ_k is the measure on \mathcal{P}_k such that $\mu_k(p) = \frac{1}{(m_k - m_{k-1})!}$ for $p \in \mathcal{P}_k$, then $\mu = \bigoplus_n \mu_n$ is the measure on \mathcal{P} . The theorem of A.M. Garsia says precisely: If $f \in L^2[0, 2\pi]$, $\{m_k\}_{k \in \mathbb{N}}$ is the above mentioned sequence and independently for each $k \in \mathbb{N}$ we permute at random the terms of the Fourier series of f whose indices are between $m_{k-1} + 1$ and m_k , then with probability μ equal to one the resulting rearranged series will converge almost everywhere. So professor Zahorski have done something which was almost impossible.

To be honest it is necessary to mention the result of R. Bilyen, R. Kallman and P. Lewis ([10]). Suppose that G is a set of all permutations of the set of natural numbers. If we put

$$d(\sigma, \sigma') = \sum_{n=1}^{\infty} 2^{-n} (d_n(\sigma, \sigma') + d_n(\sigma^{-1}\sigma'^{-1})), \quad \text{where}$$

$$d_n(\sigma, \sigma') = \frac{|\sigma_n - \sigma'_n|}{1 + |\sigma_n - \sigma'_n|},$$

then (G, d) is a Polish space. The main result in [10] says that if $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of Borel functions defined on some interval I such that $\sum_{n=1}^{\infty} f_n$ diverges almost everywhere, then the set $\{\sigma \in G : \sum_{n=1}^{\infty} f_{\sigma_n} \text{ diverges a.e.}\}$ is residual in (G, d) .

So from the point of view of Baire category it should be “easy” to find a rearrangement destroying the convergence.

N.N. Lusin in 1913 conjectured that each function in $L^2[0, 2\pi]$ has an a.e. convergent Fourier series. Professor Z. Zahorski for many years struggled with this problem and perhaps this experience helped him in finding the rearrangement. The problem of Lusin was finally solved by L. Carleson ([20]) and soon after appearing of his paper R. Hunt ([31]) was able to extend this result to all spaces $L^p[0, 2\pi]$ for $p > 1$. A.N. Kolmogorov in [32] gave an example of a function in $L^1[0, 2\pi]$ with an a.e. divergent Fourier series. This leaves only a narrow place for improving the result of Hunt. Moreover, in the example of Kolmogorov the function is in the class $L \log \log L$. Sjölin in [47] has proved that each function in the space $L \log L \log \log L$ has also an a.e. convergent Fourier series. For more informations concerning this topic see [44].

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