On the (subgroup) isomorphism problem of group rings

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Groups and their actions


Notations

\[ G \quad \text{finite group} \]
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- $U(RG)$: group of units of $RG$
- $V(RG)$: group of normalized units of $RG$, i.e.

$$V(RG) = \left\{ \sum_{g \in G} u_g g \in U(RG) : \sum_{g \in G} u_g = 1 \right\}$$
Fundamental Questions

- Given the integral group ring \( \mathbb{Z}G \). Which properties of \( G \) are determined by \( \mathbb{Z}G \) (G.Higman 1940)?
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- Given the integral group ring $\mathbb{Z}G$. Which properties of $G$ are determined by $\mathbb{Z}G$ (G. Higman 1940)?
- Given the character table $X(G)$ of a finite group $G$. What can be said over the structure of $G$?
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Let $G$ and $H$ be groups. Assume that $KG \cong KH$ for all fields $K$. Are then $G$ and $H$ isomorphic?
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- Even if one assumes that the integral group rings $\mathbb{Z}G$ and $\mathbb{Z}H$ are isomorphic it does not follow in general that $G$ and $H$ are isomorphic (M.Hertweck 1998, Annals of Math.).
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- However for some classes of finite groups Brauer’s question has an affirmative answer.

- The isomorphism problem for integral group rings is in some sense **almost** true, will be explained later.
Related Conjectures, Group Rings of Simple Groups

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Suppose that $G$ is a finite simple group. Then

$$KG \cong KH \quad \text{for all fields } K \implies G \cong H.$$ 

The proof consists of the following steps

1. Let $G$ be simple of Lie type with defining characteristic $p$ and let $K$ be a field of characteristic $p$, then $KG \cong KH$ implies that $H$ is simple.
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1. **Let $G$ be simple of Lie type with defining characteristic $p$ and let $K$ be a field of characteristic $p$, then $KG \cong KH$ implies that $H$ is simple.**

2. **Assume that $G$ is sporadic or alternating simple. If $CG$ maps onto $CH$ and $Q$ is simple then $G \cong Q$.**
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The case for the alternating groups in Step 2 has been completed by M.Nagl in his Diplomarbeit 2007.
3. By step 1 and 2 we get that

$$KG \cong KH \quad \forall K \quad \implies H \text{ is simple and } |H| = |G|.$$  

By Artin’s theorem on the order of the finite simple groups it follows that $$G \cong H$$ except \{$$G, H$$\} = \{$$A_8, A_2(4)$$\} or \{$$B_n(q), C_n(q)$$\}. 

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4 Clearly $CA_8 \not\cong CA_2(4)$.

5 $CB_n(q) \not\cong CC_n(q)$ because their smallest ordinary character degrees $\neq 1$ are different (Tiep and Zalesski).

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**Related Conjectures, Group Rings of Simple Groups II**

**Fundamental Questions**

**Group algebras of finite groups**

**Torsion subgroups of $V(ZG)$**

**SIP in Integral group rings**

**Further general results**

**Sylowlike Normalizers**

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**On the (subgroup) isomorphism problem of group rings**

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The proof of the Theorem above suggests the following.

**Modification of Brauer’s Problem 2* for simple groups**

Let $G$ be a finite simple group. Assume that $\mathbb{C}G \cong \mathbb{C}H$. Are then $G$ and $H$ isomorphic?
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**Modification of Brauer’s Problem 2* for simple groups**

Let $G$ be a finite simple group. Assume that $\mathbb{C}G \cong \mathbb{C}H$. Are then $G$ and $H$ isomorphic?

**Note** that in the case of a positive answer it follows that the rational group algebra $\mathbb{Q}G$ determines a simple group up to isomorphism.
Theorem (Hung Tong Viet 2010)

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Independently M. Nagl could show this 2009 but only for simple groups of Lie type over a prime field (with slightly different methods).
$B_n(q)$ versus $C_n(q)$

Proposition (T.Vassias 2011)

Let $p$ be the defining characteristic of $B_n(q)$ with $q$ odd. Let $K$ be a field of characteristic $p$. Then

$$KB_n(q) \not\cong KC_n(q).$$
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Let $p$ be the defining characteristic of $B_n(q)$ with $q$ odd. Let $K$ be a field of characteristic $p$. Then

$$KB_n(q) \not\cong KC_n(q).$$

Corollary

For each finite simple group of Lie type there is a field $K$ of positive characteristic such that $KG$ determines $G$ up to isomorphism.
The real problems

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Proposition (jt. work with M.Nagl)

Let $G$ be a finite simple group. Assume that $G$ has a cyclic Sylow $p$ - subgroup. Then $F_p G$ determines $G$ up to isomorphism.
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Proposition (jt. work with M.Nagl)

Let $G$ be a finite simple group. Assume that $G$ has a cyclic Sylow $p$ - subgroup. Then $F_pG$ determines $G$ up to isomorphism.

Note that each finite simple group has a cyclic Sylow subgroup for some prime $p$. 
The proof uses the theory of cyclic blocks. Because $G$ has cyclic Sylow $p$-subgroups all $p$-blocks are cyclic.
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$F_p G \cong F_p H$ implies that the composition series of the projective indecomposable modules coincide. Thus for each block the Brauer tree and the decomposition matrices are determined. This gives the degrees of the ordinary irreducible characters. Thus $C_G$ and $C_H$ are isomorphic.
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$F_p G \cong F_p H$ implies that the composition series of the projective indecomposable modules coincide. Thus for each block the Brauer tree and the decomposition matrices are determined. This gives the degrees of the ordinary irreducible characters. Thus $\mathbb{C}G$ and $\mathbb{C}H$ are isomorphic.

Now Tong Viets Theorem yields the result.
Let $U$ be a torsion subgroup of $V(\mathbb{Z}G)$

- $U$ is finite and $|U|$ divides $|G|$.
- If $U$ is central then $U < G$.
- If $|U| = |G|$ then $\mathbb{Z}G = \mathbb{Z}U$. Such a $U$ is called a group basis.
- In general a torsion subgroup is not contained in a group basis.
Description of Integral Group Rings

A typical element of $\mathbb{Z}G$ may be written as

$$\sum_{g \in G} z_g \cdot g, \ z_g \in \mathbb{Z}.$$ 

One can also consider $\mathbb{Z}G$ as order in $\mathbb{Q}G$ or $\mathbb{C}G$.

Example. Let $G = S_3 = \langle a, b; a^3, b^2, a^b = a^2 \rangle$. Then

$$\mathbb{Z}G \subset \mathbb{Z} \times \mathbb{Z} \times \left( \begin{array}{cc} \mathbb{Z} & 3\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{array} \right)$$

$$x = a_1 \times a_2 \times \left( \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right)$$ is an element of $\mathbb{Z}G$ if and only if $a_1 \equiv a_2 \mod 2$, $a_1 \equiv a_{11} \mod 3$, $a_2 \equiv a_{22} \mod 3$.

$$u = 1 \times -1 \times \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$$ corresponds to

$$u = a - a^2 - b + ab + a^2 b$$ in the other description.
Is every finite subgroup $H \leq V(\mathbb{Z}G)$ isomorphic to a subgroup of $G$?

Hertweck's counterexample (of order $2^{21} \cdot 97^{28}$) shows that in general the answer is no! Not each finite subgroup of $V(\mathbb{Z}G)$ is isomorphic to a subgroup of $G$. But for many classes of groups the isomorphism problem has a positive answer.
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The integral isomorphism problem has for $\mathbb{Z}G$ a positive solution provided $G$ is nilpotent-by-abelian (K.) or $G$ is supersoluble abelian-by-nilpotent (Roggenkamp-Zimmermann).
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$G$ is semisimple (K.-Lyons-Sandling-Teague).
Positive Results II

\[ F^* - \text{Theorem (Roggenkamp - Scott)} \]

Suppose that the generalized Fitting subgroup \( F^*(G) \) is a \( p \)-group. Then \( \mathbb{Z}G \) determines \( G \) up to isomorphism. Moreover group bases are rationally conjugate.
F* - Theorem (Roggenkamp - Scott)

Suppose that the generalized Fitting subgroup $F^*(G)$ is a $p$-group. Then $\mathbb{Z}G$ determines $G$ up to isomorphism. Moreover group bases are rationally conjugate.

Corollary

For each finite group $G$ exists an abelian extension $E = A \cdot G$ such that $\mathbb{Z}E$ determines $E$ up to isomorphism.
Thus it makes sense to ask.
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Which finite subgroups of $V(\mathbb{Z}G)$ are isomorphic to subgroups of $G$?
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Subgroup isomorphism problem (SIP)

Classify all finite groups $H$ with the following property.

Suppose that $H$ occurs as subgroup of $V(\mathbb{Z}G)$, where $G$ is finite. Then $H$ is isomorphic to a subgroup of $G$. 
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A result of Cohn - Livingstone 1963 gives a positive answer to the subgroup isomorphism problem for cyclic $p$-groups.
Fundamental questions on torsion subgroups (ctd.)

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Till 2006 this result was the only general result known on the subgroup isomorphism problem.
In particular the following important cases of the subgroup isomorphism problem are open.

- **Question A** Is a finite abelian subgroup of $V(\mathbb{Z}G)$ always isomorphic to a subgroup of $G$?
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- **Question A** Is a finite abelian subgroup of $V(\mathbb{Z}G)$ always isomorphic to a subgroup of $G$?
- **Question C** In particular, are finite cyclic subgroups of $V(\mathbb{Z}G)$ isomorphic to a subgroup of $G$?
- **Question P** Is a finite $p$ - subgroup of $V(\mathbb{Z}G)$ always isomorphic to a subgroup of $G$?
The (first) Zassenhaus Conjecture (stated 1974)

(ZC 1) Each torsion unit $u$ of $V(\mathbb{Z}G)$ is conjugate within $\mathbb{Q}G$ to an element of $G$. 

If true this provides a strong affirmative answer to Question C. It motivates in addition to SIP to ask which isomorphic torsion subgroups of $V(\mathbb{Z}G)$ are rationally conjugate.
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Known special results

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We know quite a lot in the case when $G$ is a soluble or nilpotent group, e.g.

- all torsion subgroups of $V(\mathbb{Z}G)$ are conjugate within $\mathbb{Q}G$ to a subgroup of $G$ provided $G$ is finite nilpotent (Weiss 1991).
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Remark

We know quite a lot in the case when $G$ is a soluble or nilpotent group, e.g.

- finite cyclic subgroups of $V(\mathbb{Z}G)$ are isomorphic to a subgroup of $G$ provided $G$ is finite soluble (Hertweck 2007).
- all torsion subgroups of $V(\mathbb{Z}G)$ are conjugate within $\mathbb{Q}G$ to a subgroup of $G$ provided $G$ is finite nilpotent (Weiss 1991).
The subgroup isomorphism problem has an affirmative answer if $H \cong C_2 \times C_2$. (K. 2006)
First results on non-cyclic subgroups

- The subgroup isomorphism problem has an affirmative answer if $H \cong C_2 \times C_2$. (K. 2006)

- Let $p$ be an odd prime. $V(\mathbb{Z}G)$ contains subgroups isomorphic to $C_p \times C_p$ if, and only if, $G$ has such a subgroup. (Hertweck 2007)
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Note that the proof for $p = 2$ uses the Brauer-Suzuki Theorem and is substantially different from that one for odd $p$. 
Open Problems for ZG III

Question C contains as special case the following.

**Prime graph question**

Does the prime graph of \( \mathbb{Z}G \) coincide with that one of \( G \)?

This should be seen as a first test for the Zassenhaus conjecture.
SIP for $p$ - groups

Are $p$ - subgroups of $V(\mathbb{Z}G)$ isomorphic to subgroups of $G$?
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Are $p$ - subgroups of $V(\mathbb{Z}G)$ isomorphic to subgroups of $G$?

This leads to the question whether Sylowlike theorems hold in $V(\mathbb{Z}G)$. 

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Sylowlike theorems

Is there a Sylow like theorem in the unit group of a ring? Certainly maximal $p$ - subgroups should exist and should be somehow unique.
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Typical Example.

Theorem (Abold - Plesken 1978)
In $GL(n, \mathbb{Z})$ $p$-subgroups are conjugate to a subgroup of a maximal $p$-subgroup. The conjugation takes place in $GL(n, \mathbb{Q})$. 

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Fundamental Questions

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**Theorem (Abold - Plesken 1978)**

In $GL(n, \mathbb{Z})$ $p$-subgroups are conjugate to a subgroup of a maximal $p$-subgroup. The conjugation takes place in $GL(n, \mathbb{Q})$. 
Let $G$ be a finite group. We say that in $\mathbb{Z}G$ a **strong Sylow theorem** holds provided each $p$-subgroup of $V(\mathbb{Z}G)$ is conjugate within $\mathbb{Q}G$ to a $p$-subgroup of $G$.
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It is an **open question** whether a strong Sylow theorem is valid in $\mathbb{Z}G$ for each finite group $G$. 
Evidence for a positive answer, soluble groups

a) (Roggenkamp-Scott, Weiss 1987) $\mathbb{Z}G \cong \mathbb{Z}H$, $G$ a $p$-group. Then a strong Sylow theorem holds in $V(\mathbb{Z}G)$. 
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Evidence for a positive answer, soluble groups

a) (Roggenkamp-Scott, Weiss 1987) \( \mathbb{Z} G \cong \mathbb{Z} H \), \( G \) a \( p \)-group. Then a strong Sylow theorem holds in \( V(\mathbb{Z} G) \).

b) (Dokuchaev-Juriaans 1996) \( G \) nilpotent - by - nilpotent. Then in \( \mathbb{Z} G \) a strong Sylow theorem holds.

c) (K. - Roggenkamp 1993) \( \mathbb{Z} G = \mathbb{Z} H \), \( G \) soluble. Then each \( p \)-subgroup of \( H \) is conjugate to a \( p \)-subgroup of \( G \) within \( \mathbb{Q} G \).
i.e. a strong Sylow theorem for group bases holds.
Evidence for a positive answer, outside soluble groups

d) (Dokuchaev-Juriaans-Milies 1997) Suppose that $G$ is a Frobenius group. Then a strong Sylow theorem holds for odd primes. It holds for all primes provided $G$ does not map onto $S_5$. 

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e) (A.Bächle - K. 2011) Suppose that $G$ has abelian Sylow 2-subgroups of order $\leq 8$. Then each 2-subgroup of $V(\mathbb{Z}G)$ is rationally conjugate to a subgroup of $G$. 
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f) (L.Margolis 2014) Let $G = \text{PSL}(2, q)$ with $q = 2^f$ or $q = p^f$ with $f = 1$. Then in $\mathbb{Z}G$ a strong Sylow theorem holds.

e) (A.Bächle - K. 2011) Suppose that $G$ has abelian Sylow 2-subgroups of order $\leq 8$. Then each 2-subgroup of $V(\mathbb{Z}G)$ is rationally conjugate to a subgroup of $G$. 
Theorem (A.Bächle - L.Margolis 2015)

Let $G = PSL(2, p^3)$. Then in $V(\mathbb{Z}G)$ a weak Sylow theorem holds.
Proposition (K.2014)

Suppose that $G$ has abelian Sylow $2$-subgroups then $2$-subgroups of $V(\mathbb{Z}G)$ are isomorphic to subgroups of $G$. 

Corollary

Suppose that all Sylow subgroups of odd order are cyclic or of order $p^2$ and Sylow $2$-subgroups are abelian or generalized quaternion. Then a weak Sylow theorem holds in $V(\mathbb{Z}G)$. 

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Weak Results ctd.

Proposition (K.2014)
Suppose that $G$ has abelian Sylow 2 - subgroups then 2 - subgroups of $V(\mathbb{Z}G)$ are isomorphic to subgroups of $G$.

Corollary
Suppose that all Sylow subgroups of odd order are cyclic or of order $p^2$ and Sylow 2 - subgroups are abelian or generalized quaternion. Then a weak Sylow theorem holds in $V(\mathbb{Z}G)$. 
Towards a Sylow theorem in the soluble case

In order to establish a Sylow theorem in the soluble case I believe that the following would be an important step.

Let $N$ be a normal $p$-subgroup of the soluble group $G$. Suppose further that $O_{p'}(G) = 1$. The reduction $G \rightarrow G/N$ induces a map

$$\tau : V(\mathbb{Z}G) \rightarrow V(\mathbb{Z}G/N).$$

If $U$ is torsion subgroup in $\text{Ker} \tau$, then $U$ is conjugate in $\mathbb{Z}_p(G)$ to a subgroup of $N$. 
Towards a Sylow theorem in the soluble case

In order to establish a Sylow theorem in the soluble case I believe that the following would be an important step.

Let $N$ be a normal $p$ - subgroup of the soluble group $G$. Suppose further that $O_{p'}(G) = 1$. The reduction $G \longrightarrow G/N$ induces a map

$$\tau : V(\mathbb{Z}G) \longrightarrow V(\mathbb{Z}G/N).$$

If $U$ is torsion subgroup in $Ker\tau$, then $U$ is conjugate in $\mathbb{Z}_p(G)$ to a subgroup of $N$.

A result of Hertweck shows that $U$ is rationally (i.e. in $\mathbb{Q}G$) conjugate to a subgroup of $Ker\tau$. However in many situations this is too weak.
If $U$ is an arbitrary $p$-subgroup of $V(\mathbb{Z}G)$, $G$ as above, then it is in general not true that $U$ is conjugate in $\mathbb{Z}_p(G)$ to a subgroup of $G$. 
Towards a Sylow theorem in the general case

Let $N$ be a minimal normal subgroup of $G$ which is perfect. Let $\tau$ be the induced map of $G \to G/N$ on units

$$\tau : V(\mathbb{Z}G) \to V(\mathbb{Z}G/N).$$

If $U$ is torsion subgroup in $Ker\tau$, is $U$ is conjugate in $Z_p(G)$ to a subgroup of $N$?
A Sylow theorem usually has an appendix concerning normalizers counting the number Sylow subgroups.
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Is there a good substitute in integral group rings?
The first question would be certainly whether $\mathbb{Z}G = \mathbb{Z}H$ implies that the normalizers of Sylow $p$-subgroups of $G$ and $H$ have the same size.
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**Theorem (jt. work with I.Köster)**

Let $G$ be a finite soluble group. Then for each prime $p$ the integral group ring $\mathbb{Z}G$ determines the size of the normalizer of a Sylow $p$-subgroup of $G$. 
Thank you for your attention
Finally we investigate the question whether $G$ and $H$ have the same Sylow numbers provided $\mathbb{Z}G$ and $\mathbb{Z}H$ are isomorphic as rings. In other words the question is whether the order of the normalizer of a Sylow $p$ - subgroup of $G$ is determined by $\mathbb{Z}G$. 

For each prime $p$ we denote by $n_p(G)$ the number of Sylow $p$-subgroups of $G$ and $sn(G)$ denotes the set of all Sylow numbers of $G$ for primes dividing $|G|$. The weaker question whether $sn(G)$ is determined by $X(G)$ has been raised recently by Moreto.
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The weaker question whether $sn(G)$ is determined by $X(G)$ has been raised recently by Moreto.
Moreto’s Results

Theorem (Moreto 2013)

If $sn(G) = \{a, b\}$ then $G$ is the product of two nilpotent Hall subgroups.
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Because \( X(G) \) determines whether \( G \) has nilpotent Hall \( \pi \)-subgroups (by K. - Sandling) this gives information on prime divisors of Sylow numbers.
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Because $X(G)$ determines whether $G$ has nilpotent Hall $\pi$-subgroups (by K. - Sandling) this gives information on prime divisors of Sylow numbers. Moreto’s result rescues the theorem of F.Luca that $G$ is soluble provided $sn(G) = \{a, b\}$.
**Proposition 1.** Suppose that $G$ is nilpotent - by - nilpotent. Then the character table $X(G)$ determines the Sylow numbers of $G$. 

It is easy to see that the spectral table of $G$ (i.e. $X(G)$ plus the order of the representatives of the conjugacy classes) determines $n_p(G)$ provided $G$ has cyclic Sylow $p$-subgroups.

**Proposition 2.** Suppose that $G$ is soluble. Then $ZG$ determines the Sylow numbers of $G$. The main ingredient for the proof of Theorem 2 is the $F^*$-theorem.
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Proposition 2. Suppose that $G$ is soluble. Then $\mathbb{Z}G$ determines the Sylow numbers of $G$.

The main ingredient for the proof of Theorem 2 is the $F^*$ - theorem.
Related questions, character degrees

Let $X_1(G)$ = set of the degrees of the ordinary irreducible characters of $G$. 

Huppert's Conjecture (2000):
Let $G$ and $H$ be finite groups. Assume that $G$ is simple and that $X_1(G) = X_1(H)$. Then $H \cong G \times A$, where $A$ is abelian.
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B. Huppert established his conjecture for all minimal simple groups, some other simple groups of Lie type of small Lie rank, $A_7$, $A_8$, $A_9$, $A_{10}$, for 18 of the sporadic simple groups. The remaining sporadic groups were verified by Alavi, Daneshkah, Wakefield and Tong-Viet 2011.
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- Several series of simple groups of Lie types of small Lie rank (Wakefield and Tong-Viet 2008 -2012 )

- It seems somehow to be artificial not to count (additionally) the multiplicities of the degrees?
Denote by $X_1^m(G)$ the list of the ordinary character degrees of $G$ with multiplicities.
Denote by \( X^m_1(G) \) the list of the ordinary character degrees of \( G \) with multiplicities.

Then

\[
X^m_1(G) = X^m_1(H) \text{ if and only if } CG \cong CH.
\]

Therefore the modified Brauer problem has an affirmative answer provided Hupperts conjecture holds.
Special character degrees (counted with multiplicities) have a big influence on the group structure. A typical example is
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**Theorem (Berkovich, Chillag, Herzog), Proc. AMS 1992**

The nonabelian finite groups with nonlinear characters of distinct degree are the following.

a) Extraspecial 2 - groups.

b) Frobenius groups of order $p^n(p^n - 1)$ for some prime power order $p^n$ with abelian kernel of order $p^n$ and a cyclic complement.

c) The Frobenius group of order 72 with complement isomorphic to $Q_8$. 
2-subgroups, elementary abelian subgroups

Theorem A1 (M. Hertweck - C. Höfert - K. 2008)

Let $G = \text{PSL}(2, p^f)$. Then every finite 2-subgroup of $V(\mathbb{Z}G)$ is isomorphic to a subgroup of $G$. Each elementary abelian subgroup of $V(\mathbb{Z}G)$ is isomorphic to a subgroup of $G$.

Proposition A2 (A. Bächle - K. 2008)

For any prime $p$ the elementary abelian $p$-subgroups of $V(\mathbb{Z}\text{Sz}(q))$ are isomorphic to subgroups of $\text{Sz}(q)$. If $p \in \{2, 5\}$, then this isomorphism can be taken as conjugation with a unit of $\mathbb{Q}\text{Sz}(q)$. 
Conclusion

Thank you for your attention
On the (subgroup) isomorphism problem of group rings

Kimmerle

Fundamental Questions

Group algebras of finite groups

Torsion subgroups of $V(ZG)$

SIP in Integral group rings

Further general results

Sylowlike Normalizers