On finite groups with the given set of element orders

Andrey Vasil′ev

Sobolev Institute of Mathematics,
Novosibirsk, Russia

Bedlewo, Poland, 2010

- G is a finite group
- $|G|$ is the order of G
- $\pi(G)$ is the set of prime divisors of $|G|$
- $\omega(G)$ is the spectrum of G, that is the set of its element orders

- G is a finite group
- $|G|$ is the order of G
- $\pi(G)$ is the set of prime divisors of $|G|$
- $\omega(G)$ is the spectrum of G, that is the set of its element orders
- $\mu(G)$ is the set of maximal under divisibility elements of $\omega(G)$
- $\omega(G)$ is determined by $\mu(G)$

Question 12.39, Kourovka Notebook, 1992

Is it true that a finite group and a finite simple group are isomorphic if they have the same orders and sets of element orders?

Question 12.39, Kourovka Notebook, 1992

Is it true that a finite group and a finite simple group are isomorphic if they have the same orders and sets of element orders?

This question was inspired by

Conjecture (Shi Wujie, 1987)

Every finite simple group is uniquely determined by its order and spectrum in the class of all finite groups.

W. Shi, J. Bi, H. Cao, M. Xu, 1987,...,2003

Shi's conjecture is valid for all simple groups except symplectic and orthogonal groups (more precisely, except simple groups of Lie type D_n with n even, B_n and C_n).

W. Shi, J. Bi, H. Cao, M. Xu, 1987,...,2003
Shi's conjecture is valid for all simple groups except symplectic and orthogonal groups (more precisely, except simple groups of Lie type D_n with n even, B_n and C_n).

Grechkoseeva, Mazurov and Vasil′ev, 2009
Shi's conjecture is true for remaining groups. It follows

Theorem

If L is a finite simple group, and G is a finite group with $|G| = |L|$ and $\omega(G) = \omega(L)$, then $G \simeq L$.

Put $\text{gnu}(k) =$ **group number** of k, that is the number of pairwise non-isomorphic groups of order k.

Put $\mathrm{gnu}(k)$ = **g**roup **nu**mber of k, that is the number of pairwise non-isomorphic groups of order k.

If $L = L_2(4) \simeq L_2(5) \simeq \mathrm{Alt}_5$, that is L is the smallest nonabelian simple group, then

- $\omega(L) = \{1, 2, 3, 5\}$
- $|L| = 60 = 2^2 \cdot 3 \cdot 5$
- $\mathrm{gnu}(60) = 13$

Put $\text{gnu}(k)$ = **gnu** group **nu**mber of k, that is the number of pairwise non-isomorphic groups of order k.

If $L = L_2(4) \simeq L_2(5) \simeq \text{Alt}_5$, that is L is the smallest nonabelian simple group, then

- $\omega(L) = \{1, 2, 3, 5\}$
- $|L| = 60 = 2^2 \cdot 3 \cdot 5$
- $\text{gnu}(60) = 13$

If $L = L_5(2)$, then

- $\mu(L) = \{8, 12, 14, 15, 21, 31\}$
  (recall $\omega(L)$ is the set of all divisors of elements from $\mu(L)$)
- $|L| = 9\ 999\ 360 = 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$
- $\text{gnu}(9\ 999\ 360) > 100\ 000\ 000\ 000$

The last inequality holds, since $gnu(2^{10}) = 49\ 487\ 365\ 422$ (H. U. Beshe, B. Eick, and E. A. O'Brien, 2001)

Can we omit the condition $|L| = |G|$?

Can we omit the condition $|L| = |G|$?

Fact

$L_2(4)$ and $L_5(2)$ are determined among finite groups just by $\omega(G)$

Can we omit the condition $|L| = |G|$?

Fact

$L_2(4)$ and $L_5(2)$ are determined among finite groups just by $\omega(G)$

The fact leads to the problem

Can we omit the condition $|L| = |G|$?

Fact

$L_2(4)$ and $L_5(2)$ are determined among finite groups just by $\omega(G)$

The fact leads to the problem

Problem

Which finite groups can be recognized (uniquely determined) by their spectra in the class of all finite groups?

- G is a finite group
- $h(G)$ is the number of pairwise non-isomorphic finite groups H with $\omega(H) = \omega(G)$
- G is recognizable (by spectrum) if $h(G) = 1$
- G is almost recognizable if $h(G) < \infty$
- G is non-recognizable if $h(G) = \infty$

Recognition problem

Given a finite group G, find $h(G)$. If $h(G)$ is finite, describe finite groups H with $\omega(H) = \omega(G)$.

Shi, Mazurov, 1998

If the soluble radical K of G is non-trivial then $h(G) = \infty$.

Shi, Mazurov, 1998

If the soluble radical K of G is non-trivial then $h(G) = \infty$.

There exist non-recognizable simple groups. For example, $h(L_2(9)) = \infty$, since

$$\omega(L_2(9)) = \omega(V \leftthreetimes L_2(4)),$$

where V is the elementary abelian group of order 2^4.

Shi, Mazurov, 1998

If the soluble radical K of G is non-trivial then $h(G) = \infty$.

There exist non-recognizable simple groups. For example, $h(L_2(9)) = \infty$, since

$$\omega(L_2(9)) = \omega(V \rtimes L_2(4)),$$

where V is the elementary abelian group of order 2^4.

Brandl-Shi, 1994
If $L = L_2(q)$ is a simple linear group and $q \neq 9$, then $h(L) = 1$.

Shi, Mazurov, 1998

If the soluble radical K of G is non-trivial then $h(G) = \infty$.

There exist non-recognizable simple groups. For example, $h(L_2(9)) = \infty$, since

$$\omega(L_2(9)) = \omega(V \rightthreetimes L_2(4)),$$

where V is the elementary abelian group of order 2^4.

Brandl-Shi, 1994
If $L = L_2(q)$ is a simple linear group and $q \neq 9$, then $h(L) = 1$.

Almost all groups $L_2(q)$ are recognizable.

Shi, 1987; Mazurov-Xu-Cao, 2000; Zavarnitsine-Mazurov, 2007; Mazurov-Chen, 2008; Grechkoseeva, Grechkoseeva-Vasil′ev, 2008

Let $L = L_n(q)$ be a simple linear group, $q = 2^k$, and G be a finite group. There is a number $m = m(n, q)$ such that

$$\omega(G) = \omega(L) \Leftrightarrow G \simeq L\langle\varphi\rangle,$$

where φ is a field automorphism of L and $\varphi^m = 1$.

Shi, 1987; Mazurov-Xu-Cao, 2000; Zavarnitsine-Mazurov, 2007; Mazurov-Chen, 2008; Grechkoseeva, Grechkoseeva-Vasil′ev, 2008

Let $L = L_n(q)$ be a simple linear group, $q = 2^k$, and G be a finite group. There is a number $m = m(n, q)$ such that

$$\omega(G) = \omega(L) \Leftrightarrow G \simeq L\langle\varphi\rangle,$$

where φ is a field automorphism of L and $\varphi^m = 1$.

All simple linear groups over fields of characteristic 2 are almost recognizable.

Main Conjecture

"Almost all" nonabelian simple groups are almost recognizable.

Main Conjecture

"Almost all" nonabelian simple groups are almost recognizable.

For classical groups "almost all" means "if dimension is sufficiently large".

Suppose L is a nonabelian simple group, G is a finite group with $\omega(G) = \omega(L)$. What can we say about composition factors of G?

Gruenberg-Kegel, Williams, 1981, Kondrat′ev, 1989, Vasil′ev, Vasil′ev-Vdovin, 2005

Proposition

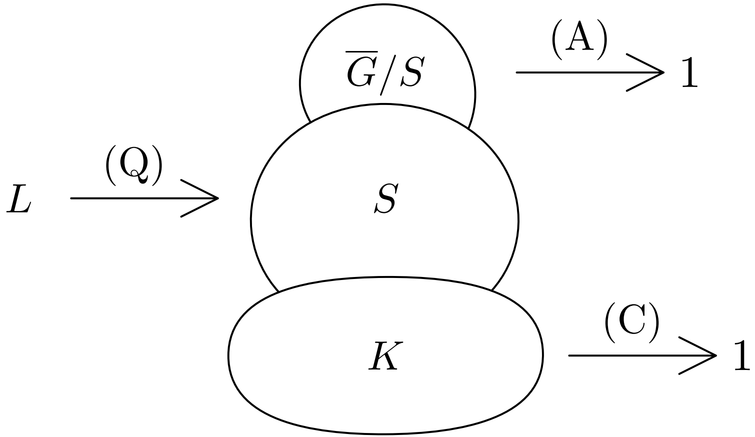
If L is not alternating and differs from $L_3(3)$, $U_3(3)$, $S_4(3)$, and G has the same spectrum (or even the same prime graph) as L, then G has exactly one nonabelian composition factor.

$$S \simeq \mathrm{Inn}(S) \leqslant G/K \leqslant \mathrm{Aut}(S)$$

K is the soluble radical of G and S is a nonabelian simple group

L is a nonabelian simple group, G is a group with $\omega(G) = \omega(L)$

$$S \leqslant \overline{G} = G/K \leqslant \mathrm{Aut}(S)$$



L is quasirecognizable if every G with $\omega(G) = \omega(L)$ has exactly one nonabelian composition factor S, and $S \simeq L$.

L is quasirecognizable if every G with $\omega(G) = \omega(L)$ has exactly one nonabelian composition factor S, and $S \simeq L$.

How can we prove the quasirecognizability of L?

L is quasirecognizable if every G with $\omega(G) = \omega(L)$ has exactly one nonabelian composition factor S, and $S \simeq L$.

How can we prove the quasirecognizability of L?
In fact, there exists one very old approach...

Approach (Sherlock Holmes, 1890)

"...when you have eliminated the impossible, whatever remains, however improbable, must be the truth..."

L is quasirecognizable if every G with $\omega(G) = \omega(L)$ has exactly one nonabelian composition factor S, and $S \simeq L$.

How can we prove the quasirecognizability of L?
In fact, there exists one very old approach...

Approach (Sherlock Holmes, 1890)

"...when you have eliminated the impossible, whatever remains, however improbable, must be the truth..."

Modern version: for as many groups L as possible we eliminate as many possibilities for S as we can

Theorem

Let L be a simple classical group over a field of characteristic p, and $L \notin \{L_2(9), L_3(3), U_3(3), U_3(5), U_5(2), S_4(3)\}$. Suppose G is a finite group with $\omega(G) = \omega(L)$ and S is the unique nonabelian composition factor of G. Then one of the following holds

- $S \simeq L$
- $L = S_4(q)$, where $q > 3$, and $S \simeq L_2(q^2)$
- $L \in \{S_6(q), O_7(q), O_8^+(q)\}$ and $S \in \{L_2(q^3), G_2(q), S_6(q), O_7(q)\}$
- $n \geq 4$, $L \in \{S_{2n}(q), O_{2n+1}(q)\}$ and $S \in \{O_{2n+1}(q), O_{2n}^-(q)\}$
- $n \geq 6$ is even, $L = O_{2n}^+(q)$ and $S \in \{S_{2n-2}(q), O_{2n-2}(q)\}$
- S is a group of Lie type over a field of characteristic $v \neq p$.

Grechkoseeva, Vasil′ev, Mazurov, 2009 (symplectic and orthogonal)
G., V., and Staroletov, 2010 (linear and unitary groups)

Theorem

Let L be a simple classical group over a field of characteristic p, and $L \notin \{L_2(9), L_3(3), U_3(3), U_3(5), U_5(2), S_4(3)\}$. Suppose G is a finite group with $\omega(G) = \omega(L)$ and S is the unique nonabelian composition factor of G. Then one of the following holds

- $S \simeq L$
- $L = S_4(q)$, where $q > 3$, and $S \simeq L_2(q^2)$
- $L \in \{S_6(q), O_7(q), O_8^+(q)\}$ and $S \in \{L_2(q^3), G_2(q), S_6(q), O_7(q)\}$
- $n \geq 4$, $L \in \{S_{2n}(q), O_{2n+1}(q)\}$ and $S \in \{O_{2n+1}(q), O_{2n}^-(q)\}$
- $n \geq 6$ is even, $L = O_{2n}^+(q)$ and $S \in \{S_{2n-2}(q), O_{2n-2}(q)\}$
- S is a group of Lie type over a field of characteristic $v \neq p$.

Grechkoseeva, Vasil′ev, Mazurov, 2009 (symplectic and orthogonal)
G., V., and Staroletov, 2010 (linear and unitary groups)
Remark. The results concerning symplectic and orthogonal groups play a significant role in proving Shi's conjecture.

Thus, the conjecture on quasirecognizability of "almost all" simple classical groups is "almost" equivalent to the following

Conjecture

Let L be a simple classical group over field of characteristic p, and S be a nonabelian composition factor of a group G with $\omega(G) = \omega(L)$. Then for "almost all" groups L the factor S is not isomorphic to a group of Lie type over field of characteristic $v \neq p$.

Thus, the conjecture on quasirecognizability of “almost all” simple classical groups is “almost” equivalent to the following

Conjecture

Let L be a simple classical group over field of characteristic p, and S be a nonabelian composition factor of a group G with $\omega(G) = \omega(L)$. Then for “almost all” groups L the factor S is not isomorphic to a group of Lie type over field of characteristic $v \neq p$.

L is a nonabelian simple group

(Q) L is quasirecognizable if every G with $\omega(G) = \omega(L)$ has exactly one nonabelian composition factor S, and $S \simeq L$.

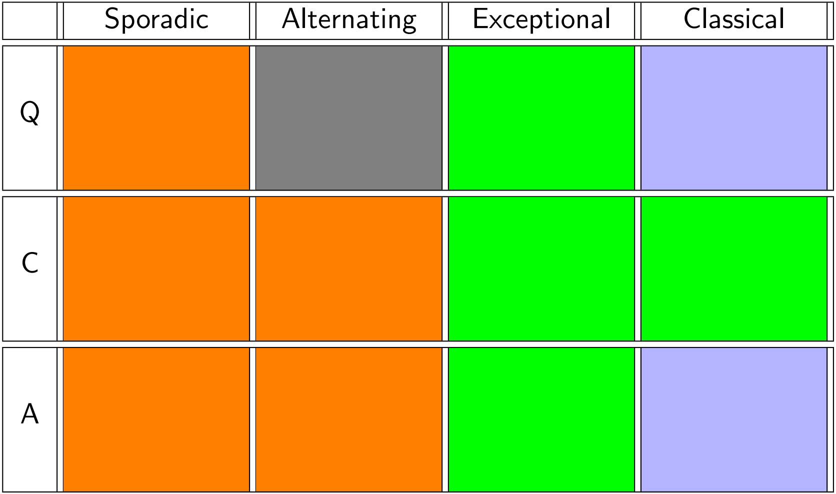
(C) L is recognizable among its coverings if for every G such that L is an homomorphic image of G the equality $\omega(G) = \omega(L)$ implies $G \simeq L$.

Note. If for L we prove (Q) and (C) then $L \leqslant G \leqslant \mathrm{Aut}(L)$.
In particular, L is almost recognizable.

(A) Describe groups G with $\omega(G) = \omega(L)$ and $L \leqslant G \leqslant \mathrm{Aut}(L)$

Solution of Recognition Problem

Achieve (Q), (C), and (A) for all nonabelian simple groups L



The problem is solved

completely mostly partially poorly

Sporadic Groups

Let L be a sporadic simple group.

Shi, 1988, . . . , Shi-Mazurov, 1998

- If $L \neq J_2$, then $h(L) = 1$.
- If $L = J_2$, then $\omega(L) = \omega(V \rtimes L_4(2))$, where V is the elementary abelian group of order 2^6, and $h(L) = \infty$.

Table

Alternating Groups

Let $L = \mathrm{Alt}_n$, $n \geqslant 5$, be a simple alternating group.

(C) Zavarnitsine, Mazurov, 1999
If G is a covering of L, then $\omega(G) \neq \omega(L)$.

(A) If $n \neq 6$, then $\mathrm{Aut}(L) = \mathrm{Sym}_n$.
If $L < G \leqslant \mathrm{Aut}(L)$, then $G = \mathrm{Sym}_n$, and $\omega(G) \neq \omega(L)$.

Table

(Q) Let L be the simple alternating group Alt_n, $n \geqslant 5$.

If $n = 6$, then $L \simeq L_2(9)$ and $h(L) = \infty$.
If $n = 10$, then there is a group G satisfying $\omega(G) = \omega(L)$ with a non-trivial soluble radical and a composition factor $S \simeq \mathrm{Alt}_5$.
If $n \neq 6, 10$ and either $n < 26$ or there is a prime in the set $\{n, n-1, n-2\}$, then $h(L) = 1$.

However, if there are no primes among the numbers $n, n-1, n-2, n-3$, nobody can even prove that a group G with $\omega(G) = \omega(L)$ has the only nonabelian composition factor.

Several years ago I. A. Vakula announced the following statement that seems provable.

If $\omega(G) = \omega(L)$ and p is the greatest prime $\leqslant n$, then among composition factors of G there is a factor $S \simeq \mathrm{Alt}_m$, where $p \leqslant m \leqslant n$.

◂ Table

Exceptional Groups of Lie Type

Let L be an exceptional group of Lie type over a field of characteristic p.

(Q) L is quasirecognizable
(unpublished for $L = E_7(q), q > 3$, which is my fault).

(C) It is sufficient to prove the following assertion.
If $L \in \{E_6^\varepsilon(q), E_7(q), {}^3D_4(q)\}$ and $G = V \rtimes L$, where V is an elementary abelian p-group, then $\omega(L) \neq \omega(G)$.

(A) It is apparently valid (and mostly proved) that $\omega(G) \neq \omega(L)$ if $L < G \leqslant \mathrm{Aut}(L)$.

Conjecture (Question 16.24 in *Kourovka Notebook*)

If L is exceptional then there are no exceptions, and $h(L) = 1$.

◂ Table

Coverings of Classical Groups

Let L be a classical group of Lie type over field of characteristic p, and G be a covering of L. Proving $\omega(G) \neq \omega(L)$ we can assume that $G = V \rtimes L$ is a semidirect product of elementary abelian r-subgroup V and the group L.

Grechkoseeva, 2010
If $r \neq p$ and the dimension of L as a matrix group is greater than 5, then $\omega(G) \neq \omega(L)$.

Zavarnitsine, 2008
If $r = p$, L is a linear or unitary group of dimension other than 4, then $\omega(G) \neq \omega(L)$.

Is it true that $\omega(G) \neq \omega(L)$, if $G = V \rtimes L$ is a semidirect product of elementary abelian p-subgroup V and a simple symplectic or orthogonal group L of dimension greater than 5?

Table

Automorphic Extensions of Classical Groups

Zavarnitsine, 2006

Let H be a connected linear algebraic group over an algebraically closed field of characteristic p and φ a surjective endomorphism of H. Given natural number r, put $H_r = C_H(\varphi^r)$. If H_r is finite for some r then φ is an automorphism of H_r of order r and

$$\omega((H_r)\langle\varphi\rangle) = \bigcup_{k|r} \frac{r}{k}\omega(H_k).$$

Application

Let $L = L_3(q)$, $q = p^n$, p an odd prime. Let G satisfy $L \leqslant G \leqslant \mathrm{Aut}(L)$ and $\omega(G) = \omega(L)$.

If $q \equiv 1 \pmod 3$ then $G = L\langle\rho^{3^i}\rangle$, where $0 \leqslant i \leqslant f$, $3^f || n$, and ρ is a field automorphism of L of order 3^f.

If $q \equiv 5, 9 \pmod{12}$ then $G = L\langle\gamma^i\rangle$, where $i = 0, 1$ and γ is a graph automorphism of L.

If $q \equiv 3, 11 \pmod{12}$ then $G = L$.

Table

Theorem

Let L be a simple classical group over a field of characteristic p, and $L \notin \{L_2(9), L_3(3), U_3(3), U_3(5), U_5(2), S_4(3)\}$. Suppose G is a finite group with $\omega(G) = \omega(L)$ and S is the unique nonabelian composition factor of G. Then one of the following holds

- $S \simeq L$
- $L = S_4(q)$, where $q > 3$, and $S \simeq L_2(q^2)$
- $L \in \{S_6(q), O_7(q), O_8^+(q)\}$ and $S \in \{L_2(q^3), G_2(q), S_6(q), O_7(q)\}$
- $n \geq 4$, $L \in \{S_{2n}(q), O_{2n+1}(q)\}$ and $S \in \{O_{2n+1}(q), O_{2n}^-(q)\}$
- $n \geq 6$ is even, $L = O_{2n}^+(q)$ and $S \in \{S_{2n-2}(q), O_{2n-2}(q)\}$
- S is a group of Lie type over a field of characteristic $v \neq p$.

Grechkoseeva, Vasil′ev, Mazurov, 2009 (symplectic and orthogonal)
G., V., and Staroletov, 2010 (linear and unitary groups)

Theorem

Let L be a simple classical group over a field of characteristic p, and $L \notin \{L_2(9), L_3(3), U_3(3), U_3(5), U_5(2), S_4(3)\}$. Suppose G is a finite group with $\omega(G) = \omega(L)$ and S is the unique nonabelian composition factor of G. Then one of the following holds

- $S \simeq L$
- $L = S_4(q)$, where $q > 3$, and $S \simeq L_2(q^2)$
- $L \in \{S_6(q), O_7(q), O_8^+(q)\}$ and $S \in \{L_2(q^3), G_2(q), S_6(q), O_7(q)\}$
- $n \geq 4$, $L \in \{S_{2n}(q), O_{2n+1}(q)\}$ and $S \in \{O_{2n+1}(q), O_{2n}^-(q)\}$
- $n \geq 6$ is even, $L = O_{2n}^+(q)$ and $S \in \{S_{2n-2}(q), O_{2n-2}(q)\}$
- S is a group of Lie type over a field of characteristic $v \neq p$.

Grechkoseeva, Vasil′ev, Mazurov, 2009 (symplectic and orthogonal)
G., V., and Staroletov, 2010 (linear and unitary groups)
Remark. The results concerning symplectic and orthogonal groups play a considerable part in proving Shi's conjecture.

Table

Quasirecognizability of Classical Groups

$L \diagdown S$	Sporadic	Alternating	Same Char	Other Char
L & U				
S & O				

The problem is solved

completely mostly partially poorly

Table

Thus, the conjecture on quasirecognizability of "almost all" simple classical groups is "almost" equivalent to the following

Conjecture

Let L be a simple classical group over field of characteristic p, and S be a nonabelian composition factor of a group G with $\omega(G) = \omega(L)$. Then for "almost all" groups L the factor S is not isomorphic to a group of Lie type over field of characteristic $v \neq p$.

Thus, the conjecture on quasirecognizability of “almost all” simple classical groups is “almost” equivalent to the following

Conjecture

Let L be a simple classical group over field of characteristic p, and S be a nonabelian composition factor of a group G with $\omega(G) = \omega(L)$. Then for “almost all” groups L the factor S is not isomorphic to a group of Lie type over field of characteristic $v \neq p$.

I strongly believe that the conjecture is true and, moreover, there is a general way to prove it for all L of sufficiently large dimension (say, for $n > 40$).

◂ Table

Simple Groups Isospectral to Soluble Groups

Lucido, Moghaddamfar, 2004
Let L be a nonabelian simple group and G be a soluble group.
$\omega(L) = \omega(G) \Rightarrow L \in \{L_3(3), U_3(3), S_4(3), \mathrm{Alt}_{10}\}$.

Simple Groups Isospectral to Soluble Groups

Lucido, Moghaddamfar, 2004
Let L be a nonabelian simple group and G be a soluble group.
$\omega(L) = \omega(G) \Rightarrow L \in \{L_3(3), U_3(3), S_4(3), \mathrm{Alt}_{10}\}$.

Theorem

Let L be a nonabelian simple group. Then a soluble group G with $\omega(G) = \omega(L)$ exists if and only if $L \in \{L_3(3), U_3(3), S_4(3)\}$.

$L_3(3)$, Mazurov (2002);
$U_3(3)$, Zinov′eva (2003);
Alt_{10}, Staroletov (2008);
$S_4(3)$, Zavarnitsine (2010).

◂ Table