On finite groups with the given set of element orders

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Bedlewo, Poland, 2010
• $G$ is a finite group
• $|G|$ is the order of $G$
• $\pi(G)$ is the set of prime divisors of $|G|$
• $\omega(G)$ is the spectrum of $G$, that is the set of its element orders
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- $|G|$ is the order of $G$
- $\pi(G)$ is the set of prime divisors of $|G|$
- $\omega(G)$ is the spectrum of $G$, that is the set of its element orders
- $\mu(G)$ is the set of maximal under divisibility elements of $\omega(G)$
- $\omega(G)$ is determined by $\mu(G)$
Is it true that a finite group and a finite simple group are isomorphic if they have the same orders and sets of element orders?
Question 12.39, Kourovka Notebook, 1992

Is it true that a finite group and a finite simple group are isomorphic if they have the same orders and sets of element orders?

This question was inspired by

Conjecture (Shi Wujie, 1987)

Every finite simple group is uniquely determined by its order and spectrum in the class of all finite groups.
Shi’s conjecture is valid for all simple groups except symplectic and orthogonal groups (more precisely, except simple groups of Lie type $D_n$ with $n$ even, $B_n$ and $C_n$).
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Grechkoseeva, Mazurov and Vasil’ev, 2009
Shi’s conjecture is true for remaining groups. It follows

**Theorem**

If $L$ is a finite simple group, and $G$ is a finite group with $|G| = |L|$ and $\omega(G) = \omega(L)$, then $G \cong L$. 
Put $\text{gnu}(k) = \text{group number of } k$, that is the number of pairwise non-isomorphic groups of order $k$. 

If $L = L_2(4) \cong L_2(5) \cong \text{Alt}_5$, that is $L$ is the smallest nonabelian simple group, then $\omega(L) = \{1, 2, 3, 5\} |L| = 60 = 2^2 \cdot 3 \cdot 5$.

If $L = L_5(2)$, then $\mu(L) = \{8, 12, 14, 15, 21, 31\}$ (recall $\omega(L)$ is the set of all divisors of elements from $\mu(L)$).

$|L| = 9,999,360 = 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$.

$\text{gnu}(9,999,360) > 100,000,000,000$.

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- $|L| = 9,999,360 = 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$
- $\text{gnu}(9,999,360) > 100,000,000,000$

The last inequality holds, since $\text{gnu}(2^{10}) = 49,487,365,422$
Can we omit the condition $|L| = |G|$?
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**Fact**

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**Problem**

Which finite groups can be recognized (uniquely determined) by their spectra in the class of all finite groups?
- $G$ is a finite group
- $h(G)$ is the number of pairwise non-isomorphic finite groups $H$ with $\omega(H) = \omega(G)$
- $G$ is recognizable (by spectrum) if $h(G) = 1$
- $G$ is almost recognizable if $h(G) < \infty$
- $G$ is non-recognizable if $h(G) = \infty$

Recognition problem

Given a finite group $G$, find $h(G)$. If $h(G)$ is finite, describe finite groups $H$ with $\omega(H) = \omega(G)$. 
Shi, Mazurov, 1998

If the soluble radical $K$ of $G$ is non-trivial then $h(G) = \infty$.
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There exist non-recognizable simple groups. For example, $h(L_2(9)) = \infty$, since

$$\omega(L_2(9)) = \omega(V \ltimes L_2(4)),$$

where $V$ is the elementary abelian group of order $2^4$. 
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Brandl-Shi, 1994

If $L = L_2(q)$ is a simple linear group and $q \neq 9$, then $h(L) = 1$. 
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If $L = L_2(q)$ is a simple linear group and $q \neq 9$, then $h(L) = 1$.

Almost all groups $L_2(q)$ are recognizable.

Let $L = L_n(q)$ be a simple linear group, $q = 2^k$, and $G$ be a finite group. There is a number $m = m(n, q)$ such that

$$\omega(G) = \omega(L) \iff G \cong L\langle \varphi \rangle,$$

where $\varphi$ is a field automorphism of $L$ and $\varphi^m = 1$. 
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All simple linear groups over fields of characteristic 2 are almost recognizable.
Main Conjecture

“Almost all” nonabelian simple groups are almost recognizable.
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For classical groups “almost all” means “if dimension is sufficiently large”.
Suppose $L$ is a nonabelian simple group, $G$ is a finite group with $\omega(G) = \omega(L)$. What can we say about composition factors of $G$?


**Proposition**

If $L$ is not alternating and differs from $L_3(3)$, $U_3(3)$, $S_4(3)$, and $G$ has the same spectrum (or even the same prime graph) as $L$, then $G$ has exactly one nonabelian composition factor.

\[ S \cong \text{Inn}(S) \leq G/K \leq \text{Aut}(S) \]

$K$ is the soluble radical of $G$ and $S$ is a nonabelian simple group.
$L$ is a nonabelian simple group, $G$ is a group with $\omega(G) = \omega(L)$

$$S \trianglelefteq \overline{G} = G/K \trianglelefteq \text{Aut}(S)$$
$L$ is quasirecognizable if every $G$ with $\omega(G) = \omega(L)$ has exactly one nonabelian composition factor $S$, and $S \cong L$. 

How can we prove the quasirecognizability of $L$?

In fact, there exists one very old approach...

Approach (Sherlock Holmes, 1890)

"...when you have eliminated the impossible, whatever remains, however improbable, must be the truth..."

Modern version: for as many groups $L$ as possible we eliminate as many possibilities for $S$ as we can.
$L$ is quasirecognizable if every $G$ with $\omega(G) = \omega(L)$ has exactly one nonabelian composition factor $S$, and $S \simeq L$.

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Theorem

Let $L$ be a simple classical group over a field of characteristic $p$, and $L \not\in \{L_2(9), L_3(3), U_3(3), U_3(5), U_5(2), S_4(3)\}$. Suppose $G$ is a finite group with $\omega(G) = \omega(L)$ and $S$ is the unique nonabelian composition factor of $G$. Then one of the following holds:

- $S \simeq L$
- $L = S_4(q)$, where $q > 3$, and $S \simeq L_2(q^2)$
- $L \in \{S_6(q), O_7(q), O_8^+(q)\}$ and $S \in \{L_2(q^3), G_2(q), S_6(q), O_7(q)\}$
- $n \geq 4$, $L \in \{S_{2n}(q), O_{2n+1}(q)\}$ and $S \in \{O_{2n+1}(q), O_{2n}^-(q)\}$
- $n \geq 6$ is even, $L = O_{2n}^+(q)$ and $S \in \{S_{2n-2}(q), O_{2n-2}(q)\}$
- $S$ is a group of Lie type over a field of characteristic $\nu \neq p$.

Grechkoseeva, Vasil’ev, Mazurov, 2009 (symplectic and orthogonal)
G., V., and Staroletov, 2010 (linear and unitary groups)
Theorem

Let $L$ be a simple classical group over a field of characteristic $p$, and $L \notin \{L_2(9), L_3(3), U_3(3), U_3(5), U_5(2), S_4(3)\}$. Suppose $G$ is a finite group with $\omega(G) = \omega(L)$ and $S$ is the unique nonabelian composition factor of $G$. Then one of the following holds

- $S \cong L$
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Remark. The results concerning symplectic and orthogonal groups play a significant role in proving Shi’s conjecture.
Thus, the conjecture on quasirecognizability of “almost all” simple classical groups is “almost” equivalent to the following

**Conjecture**

Let \( L \) be a simple classical group over field of characteristic \( p \), and \( S \) be a nonabelian composition factor of a group \( G \) with \( \omega(G) = \omega(L) \). Then for “almost all” groups \( L \) the factor \( S \) is not isomorphic to a group of Lie type over field of characteristic \( v \neq p \).
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$L$ is a nonabelian simple group

**(Q)** $L$ is quasirecognizable if every $G$ with $\omega(G) = \omega(L)$ has exactly one nonabelian composition factor $S$, and $S \simeq L$.

**(C)** $L$ is recognizable among its coverings if for every $G$ such that $L$ is an homomorphic image of $G$ the equality $\omega(G) = \omega(L)$ implies $G \simeq L$.

Note. If for $L$ we prove (Q) and (C) then $L \leq G \leq \text{Aut}(L)$. In particular, $L$ is almost recognizable.

**(A)** Describe groups $G$ with $\omega(G) = \omega(L)$ and $L \leq G \leq \text{Aut}(L)$

**Solution of Recognition Problem**

Achieve (Q), (C), and (A) for all nonabelian simple groups $L$
<table>
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<tr>
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<th>Sporadic</th>
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<th>Exceptional</th>
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The problem is solved

- **completely**
- **mostly**
- **partially**
- **poorly**
Let $L$ be a sporadic simple group.

Shi, 1988, . . . , Shi-Mazurov, 1998

- If $L \neq J_2$, then $h(L) = 1$.
- If $L = J_2$, then $\omega(L) = \omega(V \wr L_4(2))$, where $V$ is the elementary abelian group of order $2^6$, and $h(L) = \infty$. 
Let $L = \text{Alt}_n$, $n \geq 5$, be a simple alternating group.

(C) Zavarnitsine, Mazurov, 1999
If $G$ is a covering of $L$, then $\omega(G) \neq \omega(L)$.

(A) If $n \neq 6$, then $\text{Aut}(L) = \text{Sym}_n$.
If $L < G \leq \text{Aut}(L)$, then $G = \text{Sym}_n$, and $\omega(G) \neq \omega(L)$. 
(Q) Let $L$ be the simple alternating group $\text{Alt}_n$, $n \geq 5$.

If $n = 6$, then $L \simeq L_2(9)$ and $h(L) = \infty$.
If $n = 10$, then there is a group $G$ satisfying $\omega(G) = \omega(L)$ with a non-trivial soluble radical and a composition factor $S \simeq \text{Alt}_5$.
If $n \neq 6, 10$ and either $n < 26$ or there is a prime in the set 
\{ $n, n-1, n-2$ \}, then $h(L) = 1$.

However, if there are no primes among the numbers 
$n, n-1, n-2, n-3$, nobody can even prove that a group $G$ with 
$\omega(G) = \omega(L)$ has the only nonabelian composition factor.

Several years ago I. A. Vakula announced the following statement 
that seems provable.

If $\omega(G) = \omega(L)$ and $p$ is the greatest prime $\leq n$, then among 
composition factors of $G$ there is a factor $S \simeq \text{Alt}_m$, where 
$p \leq m \leq n$. 

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Exceptional Groups of Lie Type

Let $L$ be an exceptional group of Lie type over a field of characteristic $p$.

(Q) $L$ is quasirecognizable
(unpublished for $L = E_7(q), q > 3$, which is my fault).

(C) It is sufficient to prove the following assertion.
If $L \in \{E_6^e(q), E_7(q), 3D_4(q)\}$ and $G = V \rtimes L$, where $V$ is an elementary abelian $p$-group, then $\omega(L) \neq \omega(G)$.

(A) It is apparently valid (and mostly proved) that $\omega(G) \neq \omega(L)$ if $L < G \leq \text{Aut}(L)$.

Conjecture (Question 16.24 in *Kourovka Notebook*)
If $L$ is exceptional then there are no exceptions, and $h(L) = 1$. 
Coverings of Classical Groups

Let $L$ be a classical group of Lie type over field of characteristic $p$, and $G$ be a covering of $L$. Proving $\omega(G) \neq \omega(L)$ we can assume that $G = V \ltimes L$ is a semidirect product of elementary abelian $r$-subgroup $V$ and the group $L$.

Grechkoseeva, 2010
If $r \neq p$ and the dimension of $L$ as a matrix group is greater than 5, then $\omega(G) \neq \omega(L)$.

Zavarnitsine, 2008
If $r = p$, $L$ is a linear or unitary group of dimension other than 4, then $\omega(G) \neq \omega(L)$.

Is it true that $\omega(G) \neq \omega(L)$, if $G = V \ltimes L$ is a semidirect product of elementary abelian $p$-subgroup $V$ and a simple symplectic or orthogonal group $L$ of dimension greater than 5?
Let $H$ be a connected linear algebraic group over an algebraically closed field of characteristic $p$ and $\varphi$ a surjective endomorphism of $H$. Given natural number $r$, put $H_r = C_H(\varphi^r)$. If $H_r$ is finite for some $r$ then $\varphi$ is an automorphism of $H_r$ of order $r$ and

$$\omega((H_r)\langle \varphi \rangle) = \bigcup_{k|r} \frac{r}{k} \omega(H_k).$$

Application
Let $L = L_3(q)$, $q = p^n$, $p$ an odd prime. Let $G$ satisfy $L \leq G \leq \text{Aut}(L)$ and $\omega(G) = \omega(L)$.
If $q \equiv 1 \pmod{3}$ then $G = L\langle \rho^{3^i} \rangle$, where $0 \leq i \leq f$, $3^f \| n$, and $\rho$ is a field automorphism of $L$ of order $3^f$.
If $q \equiv 5, 9 \pmod{12}$ then $G = L\langle \gamma^i \rangle$, where $i = 0, 1$ and $\gamma$ is a graph automorphism of $L$.
If $q \equiv 3, 11 \pmod{12}$ then $G = L$. 
Theorem

Let $L$ be a simple classical group over a field of characteristic $p$, and $L \not\in \{L_2(9), L_3(3), U_3(3), U_3(5), U_5(2), S_4(3)\}$. Suppose $G$ is a finite group with $\omega(G) = \omega(L)$ and $S$ is the unique nonabelian composition factor of $G$. Then one of the following holds

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Remark. The results concerning symplectic and orthogonal groups play a considerable part in proving Shi’s conjecture.
Quasirecognizability of Classical Groups

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**Conjecture**

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I strongly believe that the conjecture is true and, moreover, there is a general way to prove it for all $L$ of sufficiently large dimension (say, for $n > 40$).
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Simple Groups Isospectral to Soluble Groups

Lucido, Moghaddamfar, 2004
Let $L$ be a nonabelian simple group and $G$ be a soluble group. 
$\omega(L) = \omega(G) \Rightarrow L \in \{L_3(3), U_3(3), S_4(3), \text{Alt}_{10}\}$. 
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Lucido, Moghaddamfar, 2004
Let $L$ be a nonabelian simple group and $G$ be a soluble group.
\[ \omega(L) = \omega(G) \implies L \in \{ L_3(3), U_3(3), S_4(3), \text{Alt}_{10} \} . \]

**Theorem**
Let $L$ be a nonabelian simple group. Then a soluble group $G$ with \( \omega(G) = \omega(L) \) exists if and only if \( L \in \{ L_3(3), U_3(3), S_4(3) \} \).

$L_3(3)$, Mazurov (2002);
$U_3(3)$, Zinov’eva (2003);
$\text{Alt}_{10}$, Staroletov (2008);
$S_4(3)$, Zavarnitsine (2010).