On The Width Of Verbal Subgroups In Groups Of Unitriangular Matrices

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Let $X^\ast_\infty$ denote a free group of countably infinite rank, freely generated by $X = \{x_1, x_2, \ldots\}$.

Elements of $X^\ast_\infty$ are called words. Every word $w \in X^\ast_\infty$ can be written in the form:

$$w(x_1, x_2, \ldots, x_k) = x_1^{\epsilon_1} x_2^{\epsilon_2} \ldots x_s^{\epsilon_s},$$

where $x_{ij} \in \{x_1, x_2, \ldots, x_k\} \subseteq X$ and $\epsilon_j \in \mathbb{Z} \setminus \{0\}$ for $j = 1, 2, \ldots, s$. 
**Def 1** An element $g \in G$ such that
\[ g = \nu(\underline{g}) = \nu(g_1, g_2, \ldots, g_n), \quad \underline{g} = (g_1, g_2, \ldots, g_n) \in G^n \]
is called the value of the word $\nu$ in group $G$.

- $\text{Val}(\nu, G)$ - the set of all values of the word $\nu$ in group $G$.

**Def 2** Let $W = \{w_i\}_{i \in I} \subseteq \mathbb{X}^\ast$ be a set of words and let $G$ be a group. The subgroup $V_W(G)$ generated by the set
\[ \text{Val}(W, G) = \bigcup_{w_i, i \in I} \text{Val}(w_i, G) \]
is called the verbal subgroup of $G$ generated by $W$.

- If $W = \{w\}$, we simply write $V_w(G)$ instead of $V_W(G)$.
Verbal subgroups - examples

1. The words \( c_1 = x_1, \ c_{i+1} = [c_i(x_1, \ldots, x_i), x_{i+1}] \) for \( i = 1, 2, \ldots \) are called the left-normed basic commutators.

For any group \( G \) the verbal subgroups \( V_{c_i}(G) \) constitute the lower central series \( G = \gamma_1(G) > \gamma_2(G) > \ldots \), in which

\[
\gamma_i(G) = V_{c_i}(G).
\]

**Def 3** If \( \gamma_{i+1}(G) = \{ e \} \) for some \( i \in \mathbb{N} \) then \( G \) is called nilpotent of class \( i \).
Verbal subgroups - examples

2. The commutator words

\[
\begin{align*}
    d_1(x_1) &= x_1, \\
    d_{i+1}(x_1, x_2, \ldots, x_{2i}) &= \left[ d_i(x_1, x_2, \ldots, x_{2i-1}), d_i(x_{2i-1+1}, \ldots, x_{2i}) \right]
\end{align*}
\]

for \(i = 1, 2, \ldots\) generate in every group \(G\) the verbal subgroups \(V_{d_i}(G)\), which constitute the derived series of this group:

\[
G \geq G' \geq G'' \geq G^{(3)} \geq \ldots,
\]

where \(G^{(i)} = V_{d_{i+1}}(G)\).

In particular, the derived subgroup \(G'\) of group \(G\) is its verbal subgroup \(V_{d_2}(G)\).
Properties of verbal subgroups

- invariant to endomorphisms of the group (fully invariant subgroup)
- all verbal subgroups in a group constitute the lattice
- homomorphic image of verbal subgroup is a verbal subgroup in the homomorphic image of the group
Let $W = \{w_i\}_{i \in I} \subseteq X^*_\infty$ be a set of words. Every element $v \in V_W(G)$ can be represented as a product of values of words $w_i, i \in I$.

**Def 4** *The smallest number* $n$, *such that every element* $v \in V_W(G)$ *can be represented as a product of* $n$ *values of some of the words* $w_i$ *in* $G$, $i \in I$ *is called the width of the verbal subgroup* and denoted by $\text{wid}_W(G)$.

*If no such a number exists, then $\text{wid}_W(G) = \infty$.***

- If $V_W(G) = \text{Val}(W, G)$, then $\text{wid}_W(G) = 1$.
- Verbal width depends on the set of generating words.
Width of verbal subgroups - examples

1. Let $A$ be an abelian group. For an arbitrary word $w \in X^*_\infty$

   \[ \text{wid}_w(A) = 1. \]

2. Let $S_3$ denote the symmetric group on a set of three elements and $A_3$ denote the respective alternating group. Then $V_{c_2}(S_3) = S'_3 = A_3$ and

   \[ \text{wid}_{c_2}(S_3) = 1. \]
Width of verbal subgroups - examples

3. Allambergenov, Romankov, Smirnova:

- $\text{width}_c^2(N_{n2}) = \left\lfloor \frac{n}{2} \right\rfloor$ for $n > 1$
- $\text{width}_c^2(N_{nk}) = n$ for $k \geq 3$
- $\text{width}_c^2(M_n) = n$
- $\text{width}_{X^{2k}}(N_{n2}) = 2 \left\lfloor \frac{n}{2} \right\rfloor + 1$ for $n > 1$, $k > 0$
- $\text{width}_{X^{2k+1}}(N_{n2}) = 1$

$N_{nk}$ - free nilpotent group of rank $n$ and class $k$,
$M_n$ - free metabelian group of rank $n$
Groups of unitriangular matrices
Groups of unitriangular matrices

- $K$ - an arbitrary field
- $UT_n(K)$ - the group of unitriangular $n \times n$ matrices over field $K$
Groups of unitriangular matrices

- $1_n \in UT_n(K)$ - identity matrix
- $e_{ij}$ - $n \times n$ matrix with 1 in the place $(i, j)$ and zeros elsewhere
- Every matrix $A \in UT_n(K)$ can be written as:

$$A = 1_n + \sum_{1 \leq i < j \leq n} a_{ij} e_{ij}, \quad a_{ij} \in K.$$
Groups of unitriangular matrices

The lower central series of $UT_n(K)$:

$$UT_n(K) = UT_n^0(K) > UT_n^1(K) > ... > UT_n^{n-1}(K) = \{1_n\},$$

where

$$UT_n^l(K) = \left\{ 1_n + \sum_{i<j-l \leq n} a_{i,j} e_{i,j}, \ a_{i,j} \in K \right\}, \ 0 \leq l \leq n-1.$$
We denote:

\[ P_{ij} = \{ 1_n + ae_{ij} \mid a \in K \}, \]

\[ Q_{ij} = P_{1,j}P_{1,j+1}...P_{1,n}P_{2,j}...P_{2,n}...P_{i-1,n}P_{i,j+1}...P_{i,n}. \]
Lemma 1 (Levchuk) Every characteristic subgroup $H \neq \{1\}$ in the unitriangular group $U_{T_n}(K)$ over the field $K$, $|K| > 2$, is a product of some subgroups $P_{ij}$, $i < j$, satisfying the following condition: if $P_{ij} \subseteq H$ then $Q_{ij} \subseteq H$. 
Verbal subgroups in $UT_n(K)$
**Theorem 1** Every verbal subgroup in the group $UT_n(K)$ over arbitrary field $K$ coincides with one of the terms of the lower central series of this group.

**Problem:** Given a set of words $W$, with which term of the lower central series does the verbal subgroup $V_W(UT_n(K))$ coincide?
Equalities on verbal subgroups in $UT_n(K)$

Outer-commutator words:

- outer commutator of weight 1: $w_1 = x_i$, $i \in \mathbb{N}$
- outer commutator of weight $n$: $[u_r, v_{n-r}]$, where $u_r = u(x_1, ..., x_r)$ and $v_{n-r} = v(x_{r+1}, ..., x_n)$ are outer-commutator words of weight $r$ and $n - r$ respectively

**Theorem 2** For an arbitrary field $K$, $|K| > 2$ and an outer-commutator word $w_k$ of weight $k$ we have:

$$V_{w_k}(UT_n(K)) = V_{c_k}(UT_n(K)).$$
**Theorem 3** For every field $K$, $|K| > 2$ and a power word $x^m$, where $m \in \mathbb{Z} \setminus \{0\}$ the following equalities hold:

1. If $\text{char} K = 0$ then $V_{x^m}(UT_n(K)) = UT_n(K)$;
2. If $\text{char} K = p$ and $\text{LCD}(m, p) = 1$ then
   \[ V_{x^m}(UT_n(K)) = UT_n(K); \]
3. If $\text{char} K = p$ then for every $k \in \mathbb{N}$ and $r \in \mathbb{Z}$ such that $\text{LCD}(r, p) = 1$ we have
   \[ V_{x^{p^k r}}(UT_n(K)) = V_{c^{p^k}}(UT_n(K)). \]
Width of verbal subgroups in groups of unitriangular matrices
Theorem 4  Let $\text{char} K = 0$. Then

1. $\text{wid}_{w_k}(UT_n(K)) = 1$ for every outer commutator $w_k$ of weight $k$,

2. $\text{wid}_{x_k}(UT_n(K)) = 1$ for every integer number $k$.  


Theorem 5 Let $\text{char} K = p > 0$. Then

1. $\text{wid}_{w_k}(UT_n(K)) = 1$ for every outer commutator $w_k$ of weight $k$,

2. If $k$ is an integer number, then

$$\text{wid}_{x^k}(UT_n(K)) = \begin{cases} 1 & p \nmid k \\ 2 & \text{otherwise} \end{cases}$$
Thank you.
For further reading


2. Bier A., "The width of verbal subgroups in the group of unitriangular matrices over a field", in preparation
