Powerfully nilpotent groups

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1. Introduction.
2. Presentations and growth.
3. Powerful coclass and the ancestry tree.
4. Groups of maximal powerful class.

(Joint with James Williams)

1. Introduction

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$$\{1\} = H_0 \leq H_1 \leq \cdots \leq H_n = G$$

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Definition. A powerful p-group is powerfully nilpotent if it has a powerfully central ascending chain of subgroups. The smallest length of such a chain is the powerful class of G.

The upper powerfully central series. Defined recursively by $\hat{Z}_0(G) = \{1\}$, $\hat{Z}_{n+1}(G) = \{a \in G :\ [a, x] \in \hat{Z}_n(G)^p$ for all $x \in G\}$. (Notice that $\hat{Z}_1(G) = Z(G)$).

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Theorem 1.3 (Williams). Suppose G is a powerful p-group and $N \leq G^p$ where $N \trianglelefteq G$. Then N is powerfully nilpotent.

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Theorem 2.1(T, Williams 2019). We can choose the generators $a_1, \ldots, a_r$ such that $|G| = o(a_1) \cdots o(a_r)$ and such that

$$\begin{array}{ccccccc} \langle a_1, \ldots, a_r \rangle & \geq & \langle a_1^p, a_2, \ldots, a_r \rangle & \geq & \cdots & \geq & \langle a_1^p, \ldots, a_r^p \rangle \\ & & \langle a_1^{p^2}, a_2^p, \ldots, a_r^p \rangle & \geq & \cdots & \geq & \langle a_1^{p^e}, \ldots, a_r^{p^e} \rangle = \{1\} \end{array}$$

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Powerfully nilpotent presentations . With the generators chosen as above we get relations of the form

$$\begin{aligned} [a_i, a_j] &= a_1^{m_1(i,j)} \cdots a_r^{m_r(i,j)}, \quad 1 \leq j < i \leq r \\ a_i^{n_i} &= 1, \quad 1 \leq i \leq r \end{aligned}$$

where $n_i = o(a_i)$ and where $p | m_k(i,j)$. Also $p^2 | m_k(i,j)$ when $k \leq i$.

These relations determine the structure of the group. We call such a presentation a powerfully nilpotent presentation. Conversely any powerfully nilpotent presentation gives us a powerfully nilpotent group G. We say that the presentation is consistent if $|G| = p^{n_1+\cdots+n_r}$.

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Theorem 2.2 (T, Williams 2019) Let p be an odd prime. The number of powerfully nilpotent groups of epxonenent p^2 and order p^n is $p^{\alpha n^3+o(n^3)}$, where $\alpha = \frac{9+4\sqrt{2}}{394}$

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Remark. The number of all powerful p-groups of exponent p^2 and order p^n is on the other hand $p^{\frac{2}{27}n^3+o(n^3)}$.

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Remark. Suppose H is powerfully nilpotent p-group of order $p^{n(H)}$ and powerful class $c(H)$ and that $H \to G$ where G has order $p^{n(G)}$ and powerful class $c(G)$. If $|Z(G)^p| = p^k$, then $d(G) = n(G) - c(G) = n(H) + k - (c(H) + 1) = d(H) + k - 1$. Thus

$$H \to G \Rightarrow d(H) \leq d(G)$$

with equality iff $|Z(G)^p| = p$.

Let p be a fixed prime. For any powerful p-group G we let $r = r(G)$ be the rank of G, $c = c(G)$ be the powerful class and $p^{n(G)}$, $p^{e(G)}$ be the order and exponent of G. As before the coclass is $d(G) = n(G) - c(G)$.

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Remark. As $n \leq re \leq (d+1)^2$, the two inequalities imply that there are only finitely many G with powerful coclass d.

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Theorem 4.1 (T, Williams 2019) Let G be a powerfully nilpotent group with maximal tail of rank $r \geq 2$. Then

(a) $c - 1 \leq t \leq c$ and $n - c \leq r \leq n - c + 1$.
(b) $t, c \leq 1 + r(r-1)/2$.
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Remark. If G has maximal tail then so does $G/Z(G)^p$.

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$$G = \langle y \rangle \cdot \langle b_1 \rangle \cdots \langle b_{r-1} \rangle,$$

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is powerfully central. Furthermore

(a) $[b_1, y] = b_2^p,\ [b_2, y] = b_3^p, \ldots, [b_{r-2}, y] = b_{r-1}^p, [b_{r-1}, y] = y^{p^2}$.
(b) $H = \Omega_r(G) = \langle y^p \rangle \langle b_1 \rangle \cdots \langle b_{r-1} \rangle$ p.e. G and strongly powerful.
(c) $G^{p^{r-1}} \leq Z(G)$.