Finite 2-nilpotent groups acting on compact manifolds

Groups and Their Actions International Conference

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Jordan groups

Definition

A(n infinite) group G is *Jordan* if there exists a positive integer J_G, such that every finite subgroup K of G contains a normal abelian subgroup whose index in K is at most J_G.

Example (Jordan groups, algebraic)

- $\mathrm{GL}(\mathbb{C}, n)$ for every n (Camille Jordan, 1877)
- $\mathrm{Bir}(X)$ where X is an algebraic variety for
  - $X = \mathbb{P}^2_{\mathbb{C}}$, the rank 2 Cremona group (J.-P. Serre, 2009)
  - X rationally connected (e.g. $X = \mathbb{P}^n_{\mathbb{C}}$) with $J_{\mathrm{Bir}(X)}$ depending only on $\dim X$ (Prokhorov, Shramov, 2014 + Birkar 2016)
- G connected algebraic group (with J_G depending only on $\dim G$) (Meng, Zhang, 2017)
- $\mathrm{Aut}_0(X)$ where X is a projective variety (Meng, Zhang, 2017)

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Jordan groups, smooth

Example (Jordan groups, smooth)

Diff(M) is Jordan for manifolds M such that

- M compact, $\dim(M) \leq 3$ (Bruno Zimmermann, 2014)
- Mundet i Riera (2010-2018):
  - M is the n-torus
  - M is $\mathbb{R}^n$ (acyclic manifolds)
  - M in the n-sphere (integral cohomology spheres)
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Ghys' question

Question (Ghys, < 1997)

Is Diff(M) *Jordan for every compact manifold* M*?*

Despite the positive examples above, it turned to be false:

- $\mathrm{Diff}(\mathbb{T}^2 \times \mathbb{S}^2)$ is not Jordan (Pyber-Csikós-E. Szabó, 2014) Idea: embed the Heisenberg groups $\begin{pmatrix} 1 & \mathbb{Z}_n & \mathbb{Z}_n \\ 0 & 1 & \mathbb{Z}_n \\ 0 & 0 & 1 \end{pmatrix} \subset \mathrm{Diff}(\mathbb{T}^2 \times \mathbb{S}^2)$ for infinitely many n
- Mundet i Riera (2014): higher dimensional counterexamples M_n. Idea: embed $H_{2n+1}(\mathbb{Z}_p) := \begin{pmatrix} 1 & \mathbb{Z}_p^n & \mathbb{Z}_p \\ 0 & I_n & (\mathbb{Z}_p^n)^\top \\ 0 & 0 & 1 \end{pmatrix} \subset \mathrm{Diff}(M_n)$ for certain infinite list of primes p satisfying various properties

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Main question

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For which families $\mathcal{F}$ of finite groups does there exist a compact manifold M such that $G \subseteq \mathrm{Diff}(M)$ for every $G \in \mathcal{F}$?

- Non-compact case is fully solved: exists a 4-manifold containing every finite(ly presented) group (Popov, 2013).
- Compact case: $\sup\{r(G) : G \in \mathcal{F}\} < \infty$ (Mann, Su, 1963). Here $r(G) := \max\{d(H) : H \subseteq G\}$ is the rank of G where $d(H)$ is the cardinality of a smallest generating set of H.
- Affirmative answer for:
  - for some Heisenberg groups (Pyber-Csikós-E. Szabó, Mundet i Riera, 2014)
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Note that all of these groups G are 2-nilpotent, i.e. $[G, G] \subseteq Z(G)$.

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Main result

Theorem (DSz, 2019)

For every r, there exists a compact manifold M_r such that $G \subseteq \mathrm{Diff}(M_r)$ for every finite 2-nilpotent group G of rank $\leq r$.

As an immediate corollary, we answer affirmatively a question of Mundet i Riera from 2018.

Corollary

For every n, there exists a compact manifold M_n on which every finite 2-nilpotent group G of order p^n acts faithfully via diffeomorphisms for every p.

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Overview of the group theoretic reductions

Step 1: group theoretic reductions:

1. We reduce the 2-nilpotent group G to one with cyclic centre using direct products.
2. Then further to one that is generated by 2 elements using maximal central products.
3. Finally we classify the 2-generated 2-nilpotent groups having cyclic centre.

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Reduction to 2-nilpotent groups with cyclic centre

Fix G a finite 2-nilpotent group of rank $\leq r$. (Wlog it is a p-group.)
Goal: find a manifold with a G-action depending only on r.

Lemma

There exists 2-nilpotent H_i with cyclic centre such that $G \subseteq \prod_{i=1}^{r} H_i$ and $d(H_i) \leq r$.

Proof.

- $Z(G) = \prod_{i=1}^{k} C_i$, if $k > 1$ consider $\prod_{i=1}^{k} G/C_i$ and use induction to embed G to $\prod_{i=1}^{K} G/N_i$.
- Wlog $K \leq r$ (E. Szabó): The socle of G (product of minimal normal subgroups) is $\mathbb{Z}_p^k$ for some $k \leq r$. Every non-trivial normal subgroup intersects the socle non-trivially. We select r normal subgroups $N_i \subseteq G$ from the list with trivial intersection.
- $H_i := G/N_i$, $d(H_i) \leq d(G) \leq r$. □

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Maximal central products

Definition (Maximal central product)

Given an isomorphism $\varphi : D_1 \to D_2$ for $D_i \subseteq Z(G_i)$, set

$$G_1 \curlyvee_\varphi G_2 := G_1 \times G_2 / \{(z, \varphi(z)^{-1}) : z \in D_1\},$$

the *central product* along φ amalgamating D_1 and D_2.
We call φ a maximal, if φ cannot be extended further to an isomorphism between central subgroups, and in this case the call the central product is *maximal* and denote it by $G_1 \bar{\curlyvee}_\varphi G_2$.

Lemma

$Z(G_1 \curlyvee_\varphi G_2)$ is cyclic if and only if $Z(G_1), Z(G_2)$ are both cyclic and φ is a maximal central isomorphism.

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Reduction to 2-generated groups of cyclic centre

Let H be any factor from the previous embedding.

Lemma (2-nilpotent group with cyclic centre)

There exists 2-generated 2-nilpotent groups $E_1, \ldots, E_n$ all with cyclic centre where $n \leq \lceil r/2 \rceil$, such that

$$H \cong (\ldots((E_1 \mathbin{\bar{\curlyvee}}_{\varphi_1} E_2) \mathbin{\bar{\curlyvee}}_{\varphi_2} E_3) \mathbin{\bar{\curlyvee}}_{\varphi_3} \ldots) \mathbin{\bar{\curlyvee}}_{\varphi_{n-1}} E_n,$$

and the isomorphisms class of H is independent of φ_i.

Proof.

- Find minimal generating set $\{\alpha_1, \alpha_2\} \cup S$ of E such that $H' = \langle[\alpha_1, \alpha_2]\rangle$, $[\alpha_i, s] = 1$ for all $s \in S$.
- For $H_0 := \langle S \rangle$ and $E_n := \langle \alpha, \beta \rangle$, we have $H \cong H_0 \mathbin{\bar{\curlyvee}}_{\varphi} E_n$ and use induction on d.
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- Find minimal generating set $\{\alpha_1, \alpha_2\} \cup S$ of E such that $H' = \langle [\alpha_1, \alpha_2] \rangle$, $[\alpha_i, s] = 1$ for all $s \in S$.
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- Independence follows from last part of the next Lemma. □

Reduction to 2-generated groups of cyclic centre

Let H be any factor from the previous embedding.

Lemma (2-nilpotent group with cyclic centre)

There exists 2-generated 2-nilpotent groups $E_1, \ldots, E_n$ all with cyclic centre where $n \leq \lceil r/2 \rceil$, such that

$$H \cong (\ldots((E_1 \,\bar{\curlyvee}_{\varphi_1}\, E_2) \,\bar{\curlyvee}_{\varphi_2}\, E_3) \,\bar{\curlyvee}_{\varphi_3} \ldots) \,\bar{\curlyvee}_{\varphi_{n-1}}\, E_n,$$

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2-generated groups with cyclic centre

Let E be an element of the previous decomposition.

Lemma (2-generated 2-nilpotent group with cyclic centre)

1. *$Z(E) \subseteq E$ gives $1 \to \mathbb{Z}_a \to E \to \mathbb{Z}_c \times \mathbb{Z}_c \to 1$ for some unique integers $c \mid a$,*
2. *$E = \langle \alpha, \beta, \gamma : \gamma = [\alpha, \beta], 1 = \gamma^c = [\alpha, \gamma] = [\beta, \gamma], \alpha^a = \gamma^{c_1}, \beta^c = \gamma^{c_2} \rangle$ for some integers $c_1, c_2 \mid c$. There are two types:*
   - *either $\langle \gamma \rangle = [E, E] = Z(E)$, $c = a$,*
   - *or $\langle \gamma \rangle = [E, E] \subsetneq Z(E) = \langle \alpha^c \rangle$, $c < a$, $c_1 = 1$*
3. *Every automorphism of a central subgroup can be extended to E.*

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Pull back suitable generators of E/E'. □

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Overview of the topological reductions

Step 2: Topological reductions:
Reduce from $\mathrm{Diff}(M)$ for fixed M, to $\mathrm{Diff}(L)$ for a (possibly varying) L where $L \to X$ is a line bundle over a fixed X.

Details:

1. We define the notion of *central group action on line bundles.*
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Central action

Definition

A *central action* of a finite group G on a line bundle $\pi : L \to X$ is a pair of group morphisms $(\varrho : G \to \mathrm{Diff}(L), \overline{\varrho} : G \to \mathrm{Diff}(X))$ such that

- equivariant: $\overline{\varrho}(g) \circ \pi = \pi \circ \varrho(g)$ for every $g \in G$
- linear: $\varrho(g) : L_x \to L_{\overline{\varrho}(g(x))}$ is $\mathbb{C}$-linear for every $g \in G$
- central (non-standard notion): X is compact, connected, $H^{2\bullet}(X, \mathbb{Z})$ torsion free; ϱ is injective; $\overline{\varrho}(g)$ is homotopic to the identity on X for every $g \in G$; $\mathrm{Stab}_{\overline{\varrho}}(x) = Z(G)$ for every $x \in X$.

$Z(G)$ is necessarily cyclic.

$$\begin{tikzcd}
1 \arrow[r] & Z(G) \arrow[r] \arrow[d, "{\text{faithful}, \lambda_\varrho}"'] & G \arrow[r] \arrow[d, dotted, "\varrho"] \arrow[dr, dotted, "\overline{\varrho}"] & G/Z(G) \arrow[r] \arrow[d, "{\widehat{\varrho}, \text{ free}}"] & 1 \\
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Building central actions on maximal central products

Lemma (Central product construction)

Any two central actions $\varrho_i : G_i \circlearrowright \pi_i$ induce a natural central of some $G_1 \mathbin{\bar{\curlyvee}_\varphi} G_2$.

Proof.

Quotient the natural $G_1 \times G_2$-action on $\pi_1 \boxtimes \pi_2$ by its kernel. $\square$

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Trivialising central actions

Proposition (Trivialising actions)

There exists $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that whenever ϱ is a central action of G on $\pi : L \to X$, then $G \subseteq \mathrm{Diff}(X \times \mathbb{CP}^{f(\dim X)})$.

Remark

$X \times \mathbb{CP}^{f(\dim X)}$ is compact and is independent of G (which is the main goal), although typically L very much can depend on G.

Poof Strategy:

1. Find a direct complement $\pi_\perp$ of fixed rank N with a compatible G-action using K-theory and by hand
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- By assumptions and using Atiyah—Hirzebruch theorem, we have:

$$\begin{tikzcd} K^0(X)\otimes\mathbb{Q} \arrow[d, "\cong"', "\mathrm{ch}"] & K^0(X/\overline{\varrho})\otimes\mathbb{Q} \arrow[l, "p^*"', "\cong"] \arrow[d, "\cong"', "\mathrm{ch}"] \\ H^{2\bullet}(X,\mathbb{Q}) & H^{2\bullet}(X/\overline{\varrho},\mathbb{Q}) \arrow[l, "p^*"', "\cong"] \end{tikzcd}$$

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- Enough to find a multiset $A \ni 1$ of integers of fixed size such that

$$\mathrm{ch}(\bigoplus_{a\in A}\pi^{\otimes a}) - |A| = \sum_{k=1}^{n}\sum_{a\in A}\frac{a^k}{k!}\underbrace{c_1(\pi)^k}_{\in H^{2k}(X,\mathbb{Z})} \in d\cdot H^{2\bullet}(X,\mathbb{Z}).$$

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Number theory

Lemma

For arbitrary natural n, δ, every initial sequence of integers $a_1, \ldots, a_m$ can be extended to $a_1, \ldots, a_m, a_{m+1}, \ldots, a_C$ of length $C = C(n, m)$ such that

$$\delta \mid \sum_{i=1}^{C} a_i^k \quad \forall 1 \leq k \leq n.$$

The independence of G on the manifold we are looking for translates to the independence of $C(n, |M|)$ on δ.

Proof.

- Modulo Waring problem: modulo any number, -1 can be expressed as a sum of at most W_k kth powers.
- Expand $(-\sum M + \sum_{a \in M} a) \prod_{k=2}^{n} \underbrace{(1 + a_{k,1} + \ldots a_{k,W_k})}_{k\text{th power sum} \in \delta\mathbb{Z}}$. ∎

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Proof of main theorem

Proof.

- Let G be a 2-nilpotent group of rank $\leq r$.
- By step 1: $G \subseteq \prod_{i=1}^{r} H_i$ and $H_i \cong E_{i,1} \bar{\curlyvee} E_{i,2} \bar{\curlyvee} \cdots \bar{\curlyvee} E_{i,\lceil r/2 \rceil}$ and each $E_{i,j}$ is given by a concrete presentation.
- One can construct a central action of each $E_{i,j}$ over the fixed $\mathbb{T}^2$. (Idea: the Appell–Humbert theorem describes fully the holomorphic vector bundles over $\mathbb{T}^2$ and we lift the action of $\mathbb{Z}_c^2$ on $\mathbb{T}^2$.)
- Apply the Central Product Construction $\lceil r/2 \rceil - 1$ times to get a central action of each H_i (over $\mathbb{T}^{2\lceil r/2 \rceil}$).
- The Trivialising Proposition gives a compact manifold X_r independent of H_i such that each $H_i \subseteq \mathrm{Diff}(X_r)$.
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- One can construct a central action of each $E_{i,j}$ over the fixed $\mathbb{T}^2$. (Idea: the Appell–Humbert theorem describes fully the holomorphic vector bundles over $\mathbb{T}^2$ and we lift the action of $\mathbb{Z}_c^2$ on $\mathbb{T}^2$.)
- Apply the Central Product Construction $\lceil r/2 \rceil - 1$ times to get a central action of each H_i (over $\mathbb{T}^{2\lceil r/2 \rceil}$).
- The Trivialising Proposition gives a compact manifold X_r independent of H_i such that each $H_i \subseteq \mathrm{Diff}(X_r)$.
- Then $G \subseteq \mathrm{Diff}(M_r)$ where $M_r = (X_r)^r$ is a manifold as required. $\square$

Proof of main theorem

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Thank you for your attention!