Centralizers in profinite groups

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Part 1: Introduction

Centralizers play an important role in group theory. Some results on centralizers in finite groups actually had crucial role in the establishment of finite group theory as we know it now.

Brauer-Fowler Theorem (1955): *Let G be a finite simple group and $a \in G$ an involution such that $|C_G(a)| \leq m$. Then $|G|$ is bounded by a function of m only.*

This suggested that finite simple groups could be classified by studying centralizers of involutions. Later it motivated a significant part of the classification of finite simple groups.

A group in which all centralizers are abelian is called a CA-group (sometimes CT-group).

Brauer-Suzuki-Wall theorem (1958): *Finite CA-groups are abelian, or Frobenius groups, or the groups* $\mathrm{PSL}(2, 2^m)$ *for* $m \geq 2$.

Earlier (in 1957) Suzuki showed that simple CA-groups have even order.

This had profound effect on the development of the theory of finite groups – in 1963 Feit and Thompson used Suzuki's ideas in their proof of the famous theorem that every nonabelian simple group has even order.

In a very natural way, results about finite groups have impact on the theory of locally finite groups and/or the theory of profinite groups.

Recall that locally finite groups are groups in which all finitely generated subgroups are finite. Such groups are *direct limits* of finite groups.

Profinite groups are *inverse limits* of finite groups. So there is some kind of “duality” between locally finite and profinite groups.

The study of centralizers in locally finite groups is a classical area of research. There are lots of famous results, many of which are influenced by corresponding results from the theory of finite groups.

For example, generalizing the Brauer-Fowler theorem Hartley showed that if a locally finite group G has an element x such that $C_G(x)$ is finite, then G has a locally soluble subgroup of finite index.

Khukhro showed that if the element x here has *prime order*, then G has a nilpotent subgroup of finite index.

A detailed description of locally finite CA-groups was obtained in 1998 by Yu-Fen Wu. All such groups turned out to be abelian, or Frobenius groups, or $\mathrm{PSL}(2, F)$ for a locally finite field F of characteristic 2.

On the other hand, until recently in the published literature there were almost no results on centralizers in profinite groups.

This can be easily explained by the fact that there is no natural way to reduce problems on centralizers in profinite groups to ones about finite groups.

In other words, results on centralizers in finite groups do not naturally generalize to profinite groups.

Part 2: Profinite CA-groups

In recent years infinite CA-groups attracted significant interest due to their deep relation with residually free groups. Namely, finitely generated residually free CA-groups are limit groups that played a key role in the solutions of Tarski problems. Kochloukova and Zalesski introduced a pro-p analog of limit groups which happen to be pro-p CA-groups. Further examples of pro-p CA-groups include pro-p completions of surface groups and of many 3-manifold groups (Wilkes, Zalesski, Zapata).

Thus, it would be desirable to have some structural results on profinite CA-groups.

In a recent joint work with Zapata and Zalesski we proved the following theorem.

Theorem

A profinite CA-group is virtually abelian or virtually pro-p.

Further information on the structure of profinite CA-groups is provided by the next theorems.

Theorem

Let G be a virtually abelian profinite CA-group and N is the maximal normal abelian subgroup of G. Then G/N is cyclic, or generalized quaternion, or $SL(2,3)$. In particular, G is prosoluble.

Recall that the group $SL(2,3)$ has order 24 and is isomorphic to a semidirect product of the quaternion group Q_8 by the cyclic group of order 3 which acts on Q_8 nontrivially.

Theorem

Let G be an infinite profinite CA-group.

- *If G is prosoluble and virtually pro-p, then $G = O_p(G)K$, where K is either finite cyclic or metacyclic Frobenius group.*
- *If G is not prosoluble, then $G/O_2(G)$ is almost simple.*

Here $O_p(G)$ denotes the maximal normal pro-p subgroup of G. Recall that a finite group is almost simple if it contains a nonabelian simple group and is contained in the automorphism group of that simple group.

Thus, modulo pro-p groups, we have a rather satisfying description of profinite CA-groups. Still some questions remain. For example,

Does there exist a profinite CA-group which is not prosoluble? If yes, which finite simple groups occur as sections of such groups?

Part 3: Profinite CN-groups

Profinite groups in which all centralizers are pronilpotent are called CN-groups. Finite CN-groups are a classical subject in the theory of finite groups due to the role that they have played in the proof of the Feit–Thompson theorem.

Moreover, studying such groups, Suzuki discovered the family of finite simple groups which now bears his name.

Theorem

A profinite CN-group is virtually pronilpotent.

If G is a profinite CN-group and F the maximal pronilpotent subgroup of G, we have a rather detailed description of the finite quotient G/F.

One of the following occurs.

1. G/F is cyclic.
2. G/F is a direct product of a cyclic group of odd order and a (generalized) quaternion group.
3. G/F is a Frobenius group with cyclic kernel of odd order and cyclic complement. In this case F is pro-p for some prime p.
4. G/F is isomorphic to the group $SL(2,3)$. In this case F is nilpotent and the order of F is divisible by at least two primes one of which is 2.
5. G/F is almost simple and F is a pro-2 group.

An immediate corollary of the above theorem is that the prosoluble radical in a profinite CN-group either is the whole group or is a pro-2 group. For finite CN-groups this fact was established by Suzuki in 1961.

Part 4: A characterization of virtually pro-p groups

The study of finite groups in which every element has prime-power order was iniciated by Higman in 1957. Of course, these groups are CN-groups – all centralizers in these groups are p-subgroups. Higman completely classified finite soluble groups with that property.

Nowadays also nonsoluble such groups are well-understood. In particular there are exactly eight finite simple groups with that property:

$L_2(q)$ for $q = 5; 7; 8; 9; 17$,
$L_3(4)$,
$Sz(8); Sz(32)$.

Studying profinite groups in which all centralizers are pro-p does not seem very interesting since the profinite groups in this class are just inverse limits of finite ones.

We considered profinite groups in which all centralizers $C_G(x)$ are virtually pro-p for some prime p depending on $x \in G$.

Since all finite groups have this property, the results from the theory of finite groups seem useless here.

Theorem

Let G be a profinite group such that for each $x \in G$ the centralizer $C_G(x)$ is virtually pro-p for some prime p depending on x. Then G is virtually pro-p.

Thus, if G above is infinite, then there is a *unique* prime p such that all centralizers in G are virtually pro-p.

Part 5: About proofs

Now I will describe some steps in the proof of the theorem that a profinite CA-group is virtually abelian or virtually pro-p.

Obviously, if G is pronilpotent then G is either abelian or pro-p for some prime p.

So we want to prove that G is virtually pronilpotent.

A useful technical tool is provided by the following lemma.

Lemma

Let G be a profinite CA-group having a nontrivial normal subgroup N and an infinite abelian subgroup A such that $G = NA$ and $(|A|, |N|) = 1$. Then G is abelian.

Here $|K|$ denotes the supernatural number which is the order of a profinite group K.

Recall that a profinite group G has Sylow p-subgroups for each prime $p \in \pi(G)$. If G is prosoluble, then G has a system of Sylow subgroups $P_1, P_2, \ldots,$ one for each prime dividing $|G|$, such that $P_iP_j = P_jP_i$ for all i, j. This is called a Sylow system in G.

Given such a Sylow system, let T be the intersection of the normalizers of P_i in G. The subgroup T is called system normalizer. It is known that

1. A system normalizer is always pronilpotent;
2. Any two system normalizers in a prosoluble group are conjugate;
3. $G = KT$, where $K = \gamma_\infty(G)$ is the intersection of all terms of the lower central series of G.

Given a finite soluble group G, *the Fitting height* of G is the minimal length of a normal series with nilpotent quotients. This is denoted by $h(G)$. It is clear that G is nilpotent iff $h(G) = 1$.

If G is a prosoluble group and h a positive number, we write $h(G) = h$ to mean that G has a normal series of length at most h with pronilpotent quotients.

We wish to show that if G is a prosoluble CA-group, then $h(G) \leq 5$.

First, consider the case in which G has an infinite system normalizer.

Recall that an infinite profinite group has an infinite abelian subgroup (Zelmanov, using Wilson's reduction to pro-p groups).

Suppose that G is a prosoluble CA-group having an infinite system normalizer T. Assume that T contains an infinite abelian pro-p subgroup A. Since T is a system normalizer, A normalizes a p'-Hall subgroup H of G. Now the "technical lemma" tells us that HA is abelian. In particular, G has an abelian p'-Hall subgroup.

From the corresponding result about finite groups we derive that $h(G) \leq 3$.

Now consider the case where G is a prosoluble CA-group with an infinite system normalizer T such that T does not contain infinite abelian pro-p subgroups.

Then T contains a subgroup A isomorphic to Cartesian product of finite groups of prime orders over different primes.

Since T is a system normalizer, A normalizes a Sylow p-subgroup P in some Sylow system of G. Set $B = O_{p'}(A)$. Now the "technical lemma" tells us that PB is abelian.

We see that $B \leq C_G(A) \cap C_G(P)$. Therefore $\langle A, P \rangle \leq C_G(B)$ and so PA is abelian.

This happens for each Sylow subgroup of a Sylow system. We deduce that $A \leq Z(G)$ and so G is abelian.

Thus, in all cases where T is infinite we have $h(G) \leq 3$.

In general T does not have to be infinite but in any case $h(G) \leq 5$. Why?

Write $K_1 = G$, $K_2 = \gamma_\infty(K_1)$, $K_3 = \gamma_\infty(K_2)$.

Choose a Sylow system $P_1, P_2, \dots$ in G. In a natural way this gives rise to Sylow systems $P_i \cap K_2$ and $P_i \cap K_3$ in K_2 and K_3.

Now let T_i be the system normalizer in K_i for $i = 1, 2, 3$.

If at least one T_i is infinite then $h(K_3) \leq 3$ and so $h(G) \leq 5$. Suppose that all T_i are finite. It follows that

The product $T_1T_2T_3$ is a finite CA-group of Fitting height exactly 3. This contradicts the classification of finite CA-groups!

Hence, at least one of the T_i must be infinite and so indeed $h(G) \leq 5$.

We now want to have a close look at finite groups in which all soluble subgroups have Fitting height at most 5.

Let X be the class of finite groups all of whose soluble subgroups have Fitting height at most 5. Obviously X is closed with respect to taking subgroups of its members.

It is less obvious that X is also closed with respect to taking quotients of its members.

Hence, any profinite CA-group is pro-X.

Every finite group G has a normal series each of whose factors either is soluble or is a direct product of nonabelian simple groups.

The nonsoluble length $\lambda(G)$ is defined as the minimum number of nonsoluble factors in a series of this kind.

Implicitly, the concept of nonsoluble length was used already in 1956 in the famous paper by Hall and Higman on reduction theorems for the Burnside problem.

Explicitly, the definition of nonsoluble length was introduced in a recent (2013) work of Khukhro and the speaker. We proved several new results related to this concept. In particular, we proved that

The nonsoluble length $\lambda(G)$ of a finite group G does not exceed the maximum Fitting height of soluble subgroups of G.

It follows that any group G in the class X has a normal series of length at most 35

$$1 \leq G_0 \leq G_1 \leq \cdots \leq G_{35} = G$$

all of whose quotients are either nilpotent or direct products of nonabelian finite simple groups.

From here, using the theory developed by John Wilson in his paper on compact torsion groups, we deduce that

any profinite CA-group has a normal series of length at most 35 all of whose quotients are either pronilpotent or Cartesian product of nonabelian finite simple groups.

We can prove our theorem by induction on the length of this series. The final results actually show that the series can always be chosen of length at most three.

APPENDIX: About length of profinite groups

Recall that the famous Restricted Bunside Problem can be stated as follows.

Let m, n be positive integers. Is the order of any m-generated finite group of exponent n bounded in terms of m and n only?

It seems the RBP was iniciated in a work of Grün in 1940 (Burnside actually died in 1927).

One crucial step in the eventual solution of the RBP was made in 1956 by Hall and Higman.

They showed that any finite group of exponent n has a normal series of n-bounded length all of whose quotients are either p-groups or direct products of simple groups.

The case of simple groups was handled using the classification. The case of p-groups was solved by Zelmanov using Lie algebras.

In 1983 J. S. Wilson published a work on the problem whether compact torsion groups are locally finite.

He used an earlier result of Herfort and developed a "profinite analogue" of the Hall-Higman theory to show that a compact torsion group has a finite series of closed characteristic subgroups in which each factor either is a pro-p group for some prime p or is isomorphic to a Cartesian product of isomorphic finite simple groups.

This enabled Zelmanov to prove in 1992 that compact torsion groups are locally finite.

In recent years we used Wilson's ideas on several occasions.

The most recent example is a work on "weakly Engel" groups.

Recall that a group G is called an Engel group if for every $x, g \in G$ the equation $[x, g, g, \dots, g] = 1$ holds with g repeated sufficiently many times depending on x and g.

Clearly, any locally nilpotent group is an Engel group. Wilson and Zelmanov in 1991 proved the converse for profinite groups: any Engel profinite group is locally nilpotent.

Later Medvedev extended this result to Engel compact groups.

Generalizations of Engel groups can be defined in terms of Engel sinks.

An *Engel sink* of an element g of a group G is a set $E(g)$ such that for every $x \in G$ all sufficiently long commutators $[x, g, g, \ldots, g]$ belong to $E(g)$.

Thus, G is an Engel group if and only we can choose $E(g) = \{1\}$ for all $g \in G$.

In 2018 in a joint work with E. Khukhro we considered compact groups G in which every element has a finite Engel sink and proved the following theorem.

If every element of a compact group G has a finite Engel sink, then G has a finite normal subgroup N such that G/N is locally nilpotent.

J. S. Wilson raised the question if the result remains true for compact groups in which every element has a countable sink.

Very recently (actually about a month ago) Khukhro and I answered the question in the affirmative.

If every element of a compact group G has a countable Engel sink, then G has a finite normal subgroup N such that G/N is locally nilpotent.

In the course of this work, at a crucial moment, all difficulties disappeared once we figured out a way to bound the length of our profinite group, just in the spirit of Wilson's paper of 1983.

Thank you!