Λ-trees and metric lines 000	Affine actions 0000	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines

Free actions on metric lines

Shane O Rourke

Cork Institute of Technology

10 September 2019

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Shane O Rourke Free actions on metric lines

Λ-trees and metric lines ●○○	Affine actions 0000	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines	
Let Λ (or more precisely $(\Lambda,+,\leq))$ be a (linearly) ordered abelian group.					



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For example

- $\blacksquare \Lambda = \mathbb{R}$
- $\Lambda = \mathbb{Z}^n$ (lexicographic order)



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Shane O Rourke Free actions on metric lines A-trees and metric linesAffine actionsThe case $\Lambda = \mathbb{Z}^n$ The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines $\bullet OO$ $\circ OOO$ $\circ OOOO$ $\circ OOO$ $\circ OOO$

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There is a natural notion of Λ -metric space (distances between points are non-negative elements of Λ).

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Shane O Rourke Free actions on metric lines $\begin{array}{c|c} \textbf{A-trees and metric lines} \\ \bullet \textbf{O0} \end{array} \quad \begin{array}{c} \text{Affine actions} \\ 000 \end{array} \quad \begin{array}{c} \text{The case } \Lambda = \mathbb{Z}^n \\ 0000 \end{array} \quad \begin{array}{c} \text{The case } \Lambda = \mathbb{R}^n \\ 0000 \end{array} \quad \begin{array}{c} \text{Equivariant embeddings in metric lines} \\ 000 \end{array} \\ \end{array}$

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Let
$$d(\lambda, \mu) = |\lambda - \mu| = \max\{\lambda - \mu, \mu - \lambda\}.$$

This makes Λ itself a Λ -metric space.

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Image: A math a math

 A-trees and metric lines
 Affine actions
 The case $\Lambda = \mathbb{Z}^n$ The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines

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In fact, Λ is a Λ -tree. I'll call Λ itself considered as a Λ -tree a metric line. I will look mainly at actions on metric lines today. Before we specialise to this, here are some properties of groups that act on Λ -trees...

I'll say a group is $ITF(\Lambda)$ or ITF if it admits an isometric action (without inversions) on a Λ -tree.



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Free actions on metric lines

A-trees and metric lines

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I'll say a group is ${\rm ITF}(\Lambda)$ or ${\rm ITF}$ if it admits an isometric action (without inversions) on a $\Lambda\text{-tree}.$

Some properties of ITF groups

1 torsion-free

A-trees and metric lines

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- 2 includes infinite cyclic groups
- 3 closed under free products
- 4 locally ITF implies ITF
- 5 fully residually ITF implies ITF
- 6 if G is finitely presented and ITF then G is $ITF(\mathbb{R}^n)$ for some n.
- 7 ITF(Zⁿ) groups are relatively hyperbolic with free abelian parabolic subgroups.
- 8 ITF(\mathbb{Z}^n) groups are locally relatively quasiconvex.

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A-trees and metric lines

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See survey by Kharlampovich, Miasnikov, and Serbin in IJAC (2013).

Λ-trees and metric lines	Affine actions	The case $\Lambda = \mathbb{Z}^n$	The case $\Lambda=\mathbb{R}^n$	Equivariant embeddings in metric lines
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Λ-trees and metric lines ○○●	Affine actions	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines 000

1 An ordered abelian group Λ acts freely on itself by translation, as does every subgroup of Λ .



Shane O Rourke Free actions on metric lines

A-trees and metric lines ○○●	Affine actions 0000	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines 000

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Shane O Rourke Free actions on metric lines

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A-trees and metric lines Affir	$\begin{array}{llllllllllllllllllllllllllllllllllll$	$\mathbb{Z}^n \qquad \begin{array}{c} \text{The case } \Lambda = \mathbb{R}^n \\ 00000 \end{array}$	Equivariant embeddings in metric lines

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- 2 Every torsion-free abelian group admits a linear order making it an ordered abelian group.
- **3** The group of isometries of Λ is isomorphic to $\Lambda \rtimes C_2$.

Therefore a group admits a free isometric action (without inversions) on a metric line if and only if it is torsion-free abelian.

A-trees and metric lines	Affine actions ●000	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines 000

Actions of groups on $\Lambda\text{-}{\rm trees}$ by affine automorphisms have also been studied.

 $\bullet \Lambda = \mathbb{R}:$

I. Liousse 'Actions affines sur les arbres réels'. Math. Z. (2001).

Λ-trees and metric lines 000	Affine actions ●000	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines 000

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General Λ:

SOR 'Affine actions on non-archimedean trees'. IJAC (2013).

Λ-trees and metric lines 000	Affine actions 0●00	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines 000			
We say ϕ :	$X \twoheadrightarrow X$ is	an affine aut	omorphism if	there exists			
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Λ-trees and metric lines 000	Affine actions 0●00	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines 000				
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How to generalise this to Λ -metric spaces?



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We say $\phi : X \rightarrow X$ is an affine automorphism if there exists $\alpha = \alpha_{\phi} \in \mathbb{R}$ such that $d(\phi x, \phi y) = \alpha_{\phi} d(x, y) \ \forall x, y.$

How to generalise this to Λ -metric spaces?

Note that multiplication of real numbers by positive real scalars corresponds precisely to order-preserving automorphisms of the additive real group.

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Shane O Rourke Free actions on metric lines Affine actions A for A the case $\Lambda = \mathbb{Z}^n$ The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines 0 = 0. We say $\phi : X \to X$ is an affine automorphism if there exists $\alpha_{\phi} \in \operatorname{Aut}^+(\Lambda)$ such that

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How to generalise this to Λ -metric spaces?

Note that multiplication of real numbers by positive real scalars corresponds precisely to order-preserving automorphisms of the additive real group.

So let

$$\left.\begin{array}{ccc} \alpha: \mathcal{G} & \to & \operatorname{Aut}^+(\Lambda) \\ g & \mapsto & \alpha_g \end{array}\right\}$$

be a homomorphism (Aut^+ denotes the group of order-preserving automorphisms).

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The case $\Lambda = \mathbb{Z}^n$ The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines 0 = 00 The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines 0 = 00 The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines 0 = 00 The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines 0 = 00 The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines 0 = 00 The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines 0 = 00 The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines 0 = 00 The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines 0 = 00 The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines 0 = 00 The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines 0 = 00 The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines 0 = 00 The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines 0 = 00 The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines 0 = 00 The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines 0 = 00 The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines 0 = 00 The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines 0 = 0 The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines 0 = 0 The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines 0 = 0 The case $\Lambda = \mathbb{R}^n$ the case $\Lambda = \mathbb{R}^n$

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So let

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be a homomorphism (Aut⁺ denotes the group of order-preserving automorphisms). An α -affine action of G on a Λ -tree X is an action satisfying

$$d(gx,gy) = lpha_g d(x,y) \quad \forall x,y \in X_{eg}$$
 , is the set of the

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Λ-trees and metric lines	Affine actions	The case $\Lambda = \mathbb{Z}^n$	The case $\Lambda = \mathbb{R}^n$	Equivariant embeddings in metric lines
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Some features of affine actions on general Λ -trees.

- The based length function (Lyndon length function) L_x : g → d(x, gx) can be defined and in fact determines an affine action much as in the isometric case. (The hyperbolic length function does *not* generalise easily however.)
- 2 The class ATF of groups that admit a free affine action on a Λ-tree for some Λ is closed under free products and ultraproducts.
- 3 As in the isometric case, a group G is
 - Iocally in ATF or
 - fully residually in ATF

if and only if G is in ATF.

trees and metric lines	Affine actions	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines

We will assume from now on that all affine actions on a given metric line Λ preserve the orientation of Λ .



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Λ-trees and metric lines	Affine actions	The case $\Lambda = \mathbb{Z}^n$	The case $\Lambda = \mathbb{R}^n$	Equivariant embeddings in metric lines
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We will assume from now on that all affine actions on a given metric line Λ preserve the orientation of Λ .

Which groups admit a free affine action on a metric line?



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 $\begin{array}{ccc} \Lambda \text{-trees and metric lines} & \text{Affine actions} & \text{The case } \Lambda = \mathbb{Z}^n & \text{The case } \Lambda = \mathbb{R}^n & \text{Equivariant embeddings in metric lines} \\ 000 \bullet & 000000 & 00000 & 0000 & 000 \end{array}$

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Example: Define an action of $\Gamma = \langle a, t \rangle$ on $\mathbb{Z} \times \mathbb{R}$ via

$$egin{array}{lll} {a \cdot (m,x)} &= (m, & x+1) \ {t \cdot (m,x)} &= (m+1, & rx). \end{array}$$

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Free actions on metric lines

A-trees and metric lines Affine actions The case $\Lambda = \mathbb{Z}^n$ The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines occorrection occorrectio

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This action is affine (define $\alpha : \Gamma \to \operatorname{Aut}^+(\mathbb{Z} \times \mathbb{R})$ via $\alpha_{ta} = \alpha_t : (m, x) = (m, rx)$) and free.

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A-trees and metric lines Affine actions The case $\Lambda = \mathbb{Z}^n$ The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines $000 \bullet$ 00000 00000 00000

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This action is also rigid in the sense that $g[x, y] \subseteq [x, y]$ implies g[x, y] = [x, y] (and hence g = 1 since the action is free).

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A-trees and metric lines	Affine actions	The case $\Lambda = \mathbb{Z}^n$	The case $\Lambda = \mathbb{R}^n$	Equivariant embeddings in metric lines
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• for affine automorphisms g of Λ , there exists $\nu_g \in \Lambda$ such that $g \cdot \lambda = \alpha_g(\lambda) + \nu_g$ and thus $\begin{pmatrix} \alpha_g & \nu_g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} g \cdot \lambda \\ 1 \end{pmatrix}.$



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- The group of all (order-preserving) affine automorphisms of Λ is Λ ⋊ Aut⁺(Λ), and can be represented by matrices as above.

Free actions on metric lines

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 Aut⁺(Zⁿ) ≃ UT(n, Z).

Shane O Rourke Free actions on metric lines

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•
$$\operatorname{Aut}^+(\mathbb{Z}^n) \cong \operatorname{UT}(n,\mathbb{Z}).$$

It follows that any G that admits a free affine action on Zⁿ must embed in UT(n+1, Z) ≅ Zⁿ ⋊ Aut⁺(Zⁿ).

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Λ-trees and metric lines 000	Affine actions 0000	The case $\Lambda = \mathbb{Z}^n$ $\bullet 0000000$	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines 000

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 It follows that any G that admits a free affine action on Zⁿ must embed in UT(n+1, Z) ≅ Zⁿ ⋊ Aut⁺(Zⁿ).

But the natural action of $UT(n+1,\mathbb{Z})$ on \mathbb{Z}^n is not free.

A-trees and metric lines	Affine actions	The case $\Lambda = \mathbb{Z}^n$	The case $\Lambda = \mathbb{R}^n$	Equivariant embeddings in metric lines
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$$(lpha_{ extsf{g}}-1)(\lambda) \ll |
u_{ extsf{g}}|$$
 for all λ



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A-trees and metric lines	Affine actions 0000	The case $\Lambda = \mathbb{Z}^n$ 0000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines

$$(\alpha_g - 1)(\lambda) \ll |\nu_g|$$
 for all λ

Call a matrix $A \in UT(m+1,\mathbb{Z})$ (or even $T^*(m+1,\mathbb{R})$) admissible if A = I or if the lowest non-zero entry of A - I lies in the last column and is strictly lower than any other non-zero entry.

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So $A \neq I$ is admissible if and only if A is hyperbolic and rigid.

A-trees and metric lines	Affine actions 0000	The case $\Lambda = \mathbb{Z}^n$ 0000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines

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So $A \neq I$ is admissible if and only if A is hyperbolic and rigid.

Question: Which groups admit a faithful representation as admissible matrices in $UT(m + 1, \mathbb{Z})$ for some *m*?

A-trees and metric lines Affine actions The case $\Lambda = \mathbb{Z}^n$ The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines 000 000000 00000 00000 0000

Example: Consider $x : (n_1, n_2, n_3) \mapsto (n_1, n_2 + 1, n_3)$ and $y : (n_1, n_2, n_3) \mapsto (n_1 + 1, n_2, n_3 + n_2)$.



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$$\begin{aligned} & \text{Example: Consider } x : (n_1, n_2, n_3) \mapsto (n_1, n_2 + 1, n_3) \text{ and} \\ y : (n_1, n_2, n_3) \mapsto (n_1 + 1, n_2, n_3 + n_2). \text{ In matrix form:} \\ & \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{x} \begin{pmatrix} n_3 \\ n_2 \\ n_1 \\ 1 \end{pmatrix} = \begin{pmatrix} n_3 \\ n_2 + 1 \\ n_1 \\ 1 \end{pmatrix} \\ & \underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{y} \begin{pmatrix} n_3 \\ n_2 \\ n_1 \\ 1 \end{pmatrix} = \begin{pmatrix} n_3 + n_2 \\ n_2 \\ n_1 + 1 \\ 1 \end{pmatrix} \end{aligned}$$

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The case $\Lambda = \mathbb{Z}^n$ 0000000 **Example:** Consider $x : (n_1, n_2, n_3) \mapsto (n_1, n_2 + 1, n_3)$ and $y: (n_1, n_2, n_3) \mapsto (n_1 + 1, n_2, n_3 + n_2)$. In matrix form: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_3 \\ n_2 \\ n_1 \\ 1 \end{pmatrix} = \begin{pmatrix} n_3 \\ n_2 + 1 \\ n_1 \\ 1 \end{pmatrix}$ х $\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} n_3 \\ n_2 \\ n_1 \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} n_3 + n_2 \\ n_2 \\ n_1 + \mathbf{1} \\ \mathbf{1} \end{pmatrix}$

This gives a faithful representation of $\langle x, y \rangle$ as admissible matrices in UT(4, \mathbb{Z}), and thus a free rigid affine action on \mathbb{Z}^3 .

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Λ-trees and metric lines 000	Affine actions 0000	The case $\Lambda = \mathbb{Z}^n$ 0000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines

The group $\langle x, y \rangle$ is in fact isomorphic to the discrete Heisenberg group $H_3(\mathbb{Z}) = UT(3, \mathbb{Z})$.



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Λ-trees and metric lines	Affine actions	The case $\Lambda = \mathbb{Z}^n$	The case $\Lambda = \mathbb{R}^n$	Equivariant embeddings in metric lines
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The group $\langle x, y \rangle$ is in fact isomorphic to the discrete Heisenberg group $H_3(\mathbb{Z}) = UT(3, \mathbb{Z})$.

Question: Do all unitriangular groups $UT(n, \mathbb{Z})$ admit a faithful representation as admissible matrices?

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Shane O Rourke Free actions on metric lines

Λ-trees and metric lines	Affine actions	The case $\Lambda = \mathbb{Z}^n$	The case $\Lambda = \mathbb{R}^n$	Equivariant embeddings in metric lines
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The group $\langle x, y \rangle$ is in fact isomorphic to the discrete Heisenberg group $H_3(\mathbb{Z}) = UT(3, \mathbb{Z})$.

Question: Do all unitriangular groups $UT(n, \mathbb{Z})$ admit a faithful representation as admissible matrices?

Hint (K. Dekimpe): Look at affine structures on $UT(n, \mathbb{Z})$, left symmetric algebras.

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Free actions on metric lines

Λ-trees and metric lines 000	Affine actions	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines 000			
• Consider $\mathfrak{a} = \mathfrak{ut}(n, \mathbb{Q})$ (zero upper triangular matrices):							
• Consider $\mathfrak{g} = \mathfrak{ut}(n, \mathbb{Q})$ (zero upper triangular matrices);							

Consider
$$\mathfrak{g} = \mathfrak{ut}(n, \mathbb{Q})$$
 (zero upper triangular matrices

$$[x, y] = xy - yx$$
. (Lie bracket on \mathfrak{g})



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Λ-trees and metr 000			The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embed	ldings in metric lines
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• Consider
$$\mathfrak{g} = \mathfrak{ut}(n, \mathbb{Q})$$
 (zero upper triangular matrices);

$$[x, y] = xy - yx$$
. (Lie bracket on \mathfrak{g})

If x_i has all entries equal to zero apart from those on the *i*th superdiagonal, put

$$x_i \cdot x_j = \frac{j}{i+j} [x_i, x_j].$$

Extend to a binary operation on \mathfrak{g} using bilinearity.

Free actions on metric lines

trees and metric lines	Affine actions 0000	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines 000

• Consider $\mathfrak{g} = \mathfrak{ut}(n, \mathbb{Q})$ (zero upper triangular matrices);

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If x_i has all entries equal to zero apart from those on the *i*th superdiagonal, put

$$x_i \cdot x_j = \frac{j}{i+j} [x_i, x_j].$$

Extend to a binary operation on \mathfrak{g} using bilinearity. This gives a left symmetric structure on \mathfrak{g} . That is, \cdot is a bilinear operator satisfying

1
$$[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z);$$

2 $[x, y] = x \cdot y - y \cdot x.$

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Λ-trees and metric lines 000	Affine actions	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines 000

Put m = n(n − 1)/2 and let t : g → Q^m be the linear isomorphism obtained by 'stacking the superdiagonals'.



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Λ-trees and metric lines 000	Affine actions 0000	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines

Put m = n(n − 1)/2 and let t : g → Q^m be the linear isomorphism obtained by 'stacking the superdiagonals'.
Define

$$\lambda : \mathfrak{g} \to \mathfrak{gl}(m, \mathbb{Q})$$

 $\lambda(x) : t(y) \mapsto t(x \cdot y)$

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Λ-trees and metric lines 000	Affine actions 0000	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines 000

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$$\lambda : \mathfrak{g} \to \mathfrak{gl}(m, \mathbb{Q})$$

 $\lambda(x) : t(y) \mapsto t(x \cdot y)$

Then $\lambda(x)$ is an $m \times m$ upper triangular matrix.

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-trees and metric lines 00	The case $\Lambda = \mathbb{Z}^n$ 00000000	Equivariant embeddings in metric lines

• Put
$$d\bar{\gamma}(x) = \begin{pmatrix} \lambda(x) & t(x) \\ 0 & 0 \end{pmatrix}$$
.

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Λ-trees and metric lines	Affine actions 0000	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines

• Put
$$d\bar{\gamma}(x) = \begin{pmatrix} \lambda(x) & t(x) \\ 0 & 0 \end{pmatrix}$$
.

Then $d\bar{\gamma}$ is a complete affine structure meaning that

- **1** the linear part $\lambda(x)$ of each $d\bar{\gamma}(x)$ is a nilpotent matrix;
- **2** the translation part t of $d\bar{\gamma}$ is a vector space isomorphism.

Shane O Rourke Free actions on metric lines

Λ-trees and metric lines	Affine actions	The case $\Lambda = \mathbb{Z}^n$	The case $\Lambda = \mathbb{R}^n$	Equivariant embeddings in metric lines
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• Put
$$d\bar{\gamma}(x) = \begin{pmatrix} \lambda(x) & t(x) \\ 0 & 0 \end{pmatrix}$$
.

This defines
$$d\bar{\gamma} : \mathfrak{ut}(n, \mathbb{Q}) \to \mathfrak{ut}(m+1, \mathbb{Q}).$$

Let $\varphi = \bar{\gamma} : g \mapsto \exp \cdot d\bar{\gamma} \cdot \log(g)$

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Free actions on metric lines

Λ-trees and metric lines	Affine actions	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines

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Let $\varphi = \bar{\gamma} : g \mapsto \exp \cdot d\bar{\gamma} \cdot \log(g)$

Proposition

 $\varphi : \mathrm{UT}(n, \mathbb{Q}) \to \mathrm{UT}(m+1, \mathbb{Q})$ is an injective group homomorphism with admissible image.

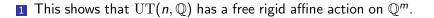
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Λ-trees and metric lines	Affine actions	The case $\Lambda = \mathbb{Z}^n$ 0000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines 000





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Λ-trees and metric lines 000	Affine actions 0000	The case $\Lambda = \mathbb{Z}^n$ 0000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines 000				
1 This s	hows that	$\mathrm{UT}(\mathit{n},\mathbb{Q})$ has	s a free rigid a	affine action on \mathbb{Q}^m .				
2 It follows that every finitely generated subgroup of $\mathrm{UT}(n,\mathbb{Q})$								
(such as $\mathrm{UT}(n,\mathbb{Z}))$ has a free rigid affine action on $\mathbb{Z}^m.$								

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Λ-trees and metric lines	Affine actions	The case $\Lambda = \mathbb{Z}^n$	The case $\Lambda = \mathbb{R}^n$	Equivariant embeddings in metric line	es		
 This shows that UT(n, Q) has a free rigid affine action on Q^m. It follows that every finitely generated subgroup of UT(n, Q) (such as UT(n, Z)) has a free rigid affine action on Z^m. 							
3 Every finitely generated torsion-free nilpotent group embeds in $UT(n, \mathbb{Z})$ for some <i>n</i> . (Jennings)							

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Free actions on metric lines

A-trees and metric lines	Affine actions	The case $\Lambda = \mathbb{Z}^n$	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines			
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(such as $\mathrm{UT}(n,\mathbb{Z}))$ has a free rigid affine action on $\mathbb{Z}^m.$							
3 Every finitely generated torsion-free nilpotent group embeds in							

UT (n,\mathbb{Z}) for some *n*. (Jennings)

Theorem

- I The groups that admit free affine actions on Zⁿ for some n are precisely finitely generated torsion-free nilpotent groups.
- Every locally residually torsion-free nilpotent group admits a free rigid affine action on a metric line.
- **3** Every free soluble group admits a free rigid affine action on a metric line.

A-trees and metric lines $\underbrace{\text{Affine actions}}_{000}$ $\underbrace{\text{The case } \Lambda = \mathbb{R}^n}_{0000}$ $\underbrace{\text{The case } \Lambda = \mathbb{R}^n}_{0000}$ $\underbrace{\text{Equivariant embeddings in metric lines}}_{000}$ Recall (once more) that BS(1, r) admits a free rigid action on $\mathbb{Z} \times \mathbb{R}$, via $a \cdot (m, x) = (m, x + 1)$ $t \cdot (m, x) = (m + 1, rx).$

This can be naturally extended to an action on $\mathbb{R} \times \mathbb{R}$



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Recall (once more) that BS(1, r) admits a free rigid action on $\mathbb{Z} \times \mathbb{R}$, via $a: (m, x) = (m, \dots, x+1)$

The case $\Lambda = \mathbb{R}^n$

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$$a \cdot (m, x) = (m, x + 1)$$

 $t \cdot (m, x) = (m + 1, rx).$

This can be naturally extended to an action on $\mathbb{R} \times \mathbb{R}$, and can be represented by admissible matrices via

$$a \mapsto \left(egin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}
ight)$$
 $t \mapsto \left(egin{array}{ccc} r & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}
ight).$

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Affine actions The case $\Lambda = \mathbb{Z}^n$ The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines **Recall** (once more) that BS(1, r) admits a free rigid action on $\mathbb{Z} \times \mathbb{R}$, via $a \cdot (m, x) = (m, x + 1)$

$$a \cdot (m, x) = (m, x + 1)$$

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This can be naturally extended to an action on $\mathbb{R} \times \mathbb{R}$, and can be represented by admissible matrices via

$$\begin{array}{c} a \mapsto \left(\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \\ t \mapsto \left(\begin{array}{ccc} r & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right). \end{array}$$

So what other (non-nilpotent) groups of upper triangular matrices admit free affine actions on \mathbb{R}^m for some m?

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Free actions on metric lines

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Λ-trees and metric lines	Affine actions 0000	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines

Let $B = T^*(n, \mathbb{R})$ denote the group of all upper triangular matrices with real entries and positive units on the diagonal.

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A-trees and metric lines Affine actions The case $\Lambda = \mathbb{Z}^n$ The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines 000

Let $B = T^*(n, \mathbb{R})$ denote the group of all upper triangular matrices with real entries and positive units on the diagonal.

Then $B = U \rtimes D^*$, where U denotes unipotent matrices and D^* denotes diagonal matrices with positive diagonal entries.

Shane O Rourke Free actions on metric lines $\begin{array}{c} \Lambda \text{-trees and metric lines} \\ 000 \end{array} \quad \begin{array}{c} \text{Affine actions} \\ 0000 \end{array} \quad \begin{array}{c} \text{The case } \Lambda = \mathbb{R}^n \\ 0000 \end{array} \quad \begin{array}{c} \text{The case } \Lambda = \mathbb{R}^n \\ 0000 \end{array} \quad \begin{array}{c} \text{Equivariant embeddings in metric lines} \\ 000 \end{array}$

Let $B = T^*(n, \mathbb{R})$ denote the group of all upper triangular matrices with real entries and positive units on the diagonal.

Then $B = U \rtimes D^*$, where U denotes unipotent matrices and D^* denotes diagonal matrices with positive diagonal entries.

Theorem

The group $T^*(n, \mathbb{R})$ admits an embedding in $T^*(n + m + 1, \mathbb{R})$ with admissible image. Thus $T^*(n, \mathbb{R})$ admits a free rigid affine action on \mathbb{R}^{n+m} (considered as an \mathbb{R}^{n+m} -tree). Let $B = T^*(n, \mathbb{R})$ denote the group of all upper triangular matrices with real entries and positive units on the diagonal.

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Theorem

The group $T^*(n, \mathbb{R})$ admits an embedding in $T^*(n + m + 1, \mathbb{R})$ with admissible image. Thus $T^*(n, \mathbb{R})$ admits a free rigid affine action on \mathbb{R}^{n+m} (considered as an \mathbb{R}^{n+m} -tree).

The proof loosely follows an argument of John Milnor (see the proof of Theorem 1.2 in 'On Fundamental Groups of Complete Affinely Flat Manifolds' (Adv. Math. 1977)).

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A-trees and metric lines	Affine actions 0000	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines

Example: n = 3

A typical element of $T^*(3,\mathbb{R})$ is expressible in the form *ud* where

.

$$u = \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \text{ and } d = \begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{pmatrix}$$

Free actions on metric lines

A-trees and metric lines	Affine actions 0000	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines

Example: n = 3A typical element of $T^*(3,\mathbb{R})$ is expressible in the form *ud* where $u = \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \text{ and } d = \begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{pmatrix}.$ Now $\varphi(u) = \begin{pmatrix} 1 & -x/2 & y/2 & z - xy/2 \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ \hline 0 & 0 & 0 & 1 \end{pmatrix}$ so that $\varphi_0(u) = \begin{pmatrix} 1 & -x/2 & y/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $b(u) = \begin{pmatrix} z - xy/2 \\ y \\ x \end{pmatrix}$ where φ is our admissible embedding of $UT(n, \mathbb{Q})$ in $UT(m+1, \mathbb{Q})$.

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Affine actions Affine actions The case $\Lambda = \mathbb{Z}^n$ The case $\Lambda = \mathbb{R}^n$ Equivariant embeddings in metric lines occord to $\mathcal{A}^* = \begin{pmatrix} r/t & 0 & 0 \\ 0 & r/s & 0 \\ 0 & 0 & s/t \end{pmatrix}$ so that $\bar{\varphi}(ud) = \bar{\varphi}(u)\bar{\varphi}(d)$

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Affine actions The case
$$\Lambda = 2^n$$
 The case $\Lambda = 1^n$ Equivariant embeddings in metric lines
 $\phi(u) = \begin{pmatrix} r/t & 0 & 0 \\ 0 & r/s & 0 \\ 0 & 0 & s/t \end{pmatrix}$ so that $\bar{\varphi}(ud) = \bar{\varphi}(u)\bar{\varphi}(d)$ where
 $\vec{\varphi}(u) = \begin{pmatrix} 1 & -x/2 & y/2 & 0 & 0 & 0 & z - xy/2 \\ 0 & 1 & 0 & 0 & 0 & y \\ 0 & 0 & 1 & 0 & 0 & 0 & x \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

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Λ-trees and metric line	es Affine 0000		The ca				he case $\Lambda = \mathbb{R}^n$ 0000	Equivariant embeddings in metri 000	
Also d* =	$= \begin{pmatrix} r_{j} \\ 0 \\ 0 \end{pmatrix}$	/t () r _/) () ('s () s _/)) (t)	s	o th	hat $ar{arphi}(\mathit{ud})$	$=ar{arphi}(u)ar{arphi}(d)$ where	
	/1 -	-x/2	y/2	0	0	0	z - xy	2 \	
	0	1	0	0	0	0	y y		
$ar{arphi}(u) =$	0	0	1	0	0	0	x		
$\bar{\varphi}(u) =$	0	0	0	1	0	0	0	and	
	0	0	0	0	1	0	0		
	0	0	0	0	0	1	0		
	$\sqrt{0}$	0	0	0	0	0	1	—)	
$ar{arphi}(d) =$	$\int r/t$	0	0	0	0	0	0		
	0	r/s	0	0	0	0	0		
	0	0	s/t	0	0	0	0		
$ar{arphi}(d) =$	0	0	0	1	0	0	$\log(r)$		
	0	0	0	0	1	0	$\log(s)$		
	0	0	0	0	0	1	$\log(t)$		
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Λ-trees and metric lines	Affine actions	The case $\Lambda = \mathbb{Z}^n$	The case $\Lambda = \mathbb{R}^n$	Equivariant embeddings in metric lines
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We can also show that the wreath product $\Lambda_1 \wr \Lambda_2$ of two ordered abelian groups admits a free rigid affine action on a metric line.

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Free actions on metric lines

A-trees and metric lines	Affine actions	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines 000

We can also show that the wreath product $\Lambda_1 \wr \Lambda_2$ of two ordered abelian groups admits a free rigid affine action on a metric line. More generally $\Lambda_1 \wr G$ admits such an action where G admits a free rigid affine action on a metric line.

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Free actions on metric lines

Λ-trees and metric lines	Affine actions	The case $\Lambda = \mathbb{Z}^n$	The case $\Lambda = \mathbb{R}^n$	Equivariant embeddings in metric lines
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We can also show that the wreath product $\Lambda_1 \wr \Lambda_2$ of two ordered abelian groups admits a free rigid affine action on a metric line. More generally $\Lambda_1 \wr G$ admits such an action where G admits a free rigid affine action on a metric line.

Hence an iterated wreath product of ordered abelian groups admits such an action.

Λ-trees a 000	nd metric lines	Affine actions	The case $\Lambda = \mathbb{Z}^n$	The case $\Lambda = \mathbb{R}^n$	Equivariant embeddings in metric lines ●୦୦
F		e following ght-orderal		ent for a grou	pG.

2 *G* admits a free order-preserving action on a linearly ordered set.



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Λ-trees and metric lines	Affine actions	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines •••

Fact 1: The following are equivalent for a group *G*.

- **1** *G* is right-orderable;
- 2 *G* admits a free order-preserving action on a linearly ordered set.
- Fact 2: The following are equivalent for a group G.
 - **1** *G* is orderable;
 - 2 *G* admits a free, rigid, order-preserving action on a linearly ordered set;
 - **3** *G* admits a faithful order-preserving action on a linearly ordered set.

Λ-trees and metric lines	Affine actions	The case $\Lambda = \mathbb{Z}^n$	The case $\Lambda = \mathbb{R}^n$	Equivariant embeddings in metric lines
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 - **1** *G* is orderable;
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 - **3** *G* admits a faithful order-preserving action on a linearly ordered set.

Question: Suppose G has an order-preserving action on a linearly ordered set X. Under what conditions does X admit an equivariant embedding in a metric line \hat{X} equipped with (a) an affine action? (b) a rigid affine action?

A-trees and metric lines	Affine actions 0000	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines 000

We are working towards:

Let G be a group equipped with an order-preserving action on a linearly ordered set X. Then there exists an ordered abelian group Λ , an affine action of G on Λ and an equivariant embedding of X in Λ .

Shane O Rourke Free actions on metric lines

A-trees and metric lines	Affine actions 0000	The case $\Lambda = \mathbb{Z}^n$ 0000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines 000

We are working towards:

- Let G be a group equipped with an order-preserving action on a linearly ordered set X. Then there exists an ordered abelian group Λ , an affine action of G on Λ and an equivariant embedding of X in Λ .
- 2 Let G be a group equipped with a rigid order-preserving action on a linearly ordered set X. Then there exists an ordered abelian group Λ, a rigid affine action of G on Λ and an equivariant embedding of X in Λ.

Λ-trees and metric lines	Affine actions	The case $\Lambda = \mathbb{Z}^n$ 00000000	The case $\Lambda = \mathbb{R}^n$ 00000	Equivariant embeddings in metric lines

Dziękuję bardzo!

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