Free actions on metric lines

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In fact, Λ is a Λ-tree. I'll call Λ itself considered as a Λ-tree a metric line. I will look mainly at actions on metric lines today.
Before we specialise to this, here are some properties of groups that act on Λ-trees...

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Some properties of ITF groups

1. torsion-free
2. includes infinite cyclic groups
3. closed under free products
4. locally ITF implies ITF
5. fully residually ITF implies ITF
6. if G is finitely presented and ITF then G is $\mathrm{ITF}(\mathbb{R}^n)$ for some n.
7. $\mathrm{ITF}(\mathbb{Z}^n)$ groups are relatively hyperbolic with free abelian parabolic subgroups.
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See survey by Kharlampovich, Miasnikov, and Serbin in IJAC (2013).

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Therefore a group admits a free isometric action (without inversions) on a metric line if and only if it is torsion-free abelian.

Actions of groups on Λ-trees by affine automorphisms have also been studied.

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- General Λ:
  SOR 'Affine actions on non-archimedean trees'.
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be a homomorphism (Aut^+ denotes the group of order-preserving automorphisms).

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be a homomorphism (Aut^+ denotes the group of order-preserving automorphisms). An α-affine action of G on a Λ-tree X is an action satisfying

$$d(gx, gy) = \alpha_g d(x, y) \quad \forall x, y \in X$$

Some features of affine actions on general Λ-trees.

1. The based length function (Lyndon length function) $L_x : g \mapsto d(x, gx)$ can be defined and in fact determines an affine action much as in the isometric case. (The hyperbolic length function does *not* generalise easily however.)
2. The class ATF of groups that admit a free affine action on a Λ-tree for some Λ is closed under free products and ultraproducts.
3. As in the isometric case, a group G is
   - locally in ATF or
   - fully residually in ATF

   if and only if G is in ATF.

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$$\begin{aligned} a \cdot (m, x) &= (m, \quad x + 1) \\ t \cdot (m, x) &= (m + 1, \quad rx). \end{aligned}$$

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This action is also rigid in the sense that $g[x, y] \subseteq [x, y]$ implies $g[x, y] = [x, y]$ (and hence $g = 1$ since the action is free).

Note that

- for affine automorphisms g of Λ, there exists $\nu_g \in \Lambda$ such that $g \cdot \lambda = \alpha_g(\lambda) + \nu_g$ and thus
$$\begin{pmatrix} \alpha_g & \nu_g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} g \cdot \lambda \\ 1 \end{pmatrix}.$$

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- $\mathrm{Aut}^+(\mathbb{Z}^n) \cong \mathrm{UT}(n, \mathbb{Z})$.
- It follows that any G that admits a free affine action on $\mathbb{Z}^n$ must embed in $\mathrm{UT}(n+1, \mathbb{Z}) \cong \mathbb{Z}^n \rtimes \mathrm{Aut}^+(\mathbb{Z}^n)$.

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But the natural action of $\mathrm{UT}(n+1, \mathbb{Z})$ on $\mathbb{Z}^n$ is not free.

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Call a matrix $A \in \mathrm{UT}(m+1, \mathbb{Z})$ (or even $T^*(m+1, \mathbb{R})$) admissible if $A = I$ or if the lowest non-zero entry of $A - I$ lies in the last column and is strictly lower than any other non-zero entry.

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So $A \neq I$ is admissible if and only if A is hyperbolic and rigid.

Question: Which groups admit a faithful representation as admissible matrices in $\mathrm{UT}(m+1, \mathbb{Z})$ for some m?

Example: Consider $x : (n_1, n_2, n_3) \mapsto (n_1, n_2 + 1, n_3)$ and $y : (n_1, n_2, n_3) \mapsto (n_1 + 1, n_2, n_3 + n_2)$.

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$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{x} \begin{pmatrix} n_3 \\ n_2 \\ n_1 \\ 1 \end{pmatrix} = \begin{pmatrix} n_3 \\ n_2 + 1 \\ n_1 \\ 1 \end{pmatrix}$$

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This gives a faithful representation of $\langle x, y \rangle$ as admissible matrices in $\mathrm{UT}(4, \mathbb{Z})$, and thus a free rigid affine action on $\mathbb{Z}^3$.

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The group $\langle x, y\rangle$ is in fact isomorphic to the discrete Heisenberg group $H_3(\mathbb{Z}) = \mathrm{UT}(3, \mathbb{Z})$.

Question: Do all unitriangular groups $\mathrm{UT}(n, \mathbb{Z})$ admit a faithful representation as admissible matrices?

Hint (K. Dekimpe): Look at affine structures on $\mathrm{UT}(n, \mathbb{Z})$, left symmetric algebras.

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  $[x, y] = xy - yx$. (Lie bracket on $\mathfrak{g}$)

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  Extend to a binary operation on $\mathfrak{g}$ using bilinearity. This gives a left symmetric structure on $\mathfrak{g}$. That is, $\cdot$ is a bilinear operator satisfying

  1. $[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z)$;
  2. $[x, y] = x \cdot y - y \cdot x$.

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Then $\lambda(x)$ is an $m \times m$ upper triangular matrix.

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  Then $d\bar{\gamma}$ is a complete affine structure meaning that
  1. the linear part $\lambda(x)$ of each $d\bar{\gamma}(x)$ is a nilpotent matrix;
  2. the translation part t of $d\bar{\gamma}$ is a vector space isomorphism.

- Put $d\bar{\gamma}(x) = \begin{pmatrix} \lambda(x) & t(x) \\ 0 & 0 \end{pmatrix}$.

  This defines $d\bar{\gamma} : \mathfrak{ut}(n, \mathbb{Q}) \to \mathfrak{ut}(m+1, \mathbb{Q})$.
- Let $\varphi = \bar{\gamma} : g \mapsto \exp \cdot d\bar{\gamma} \cdot \log(g)$

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Proposition

$\varphi : \mathrm{UT}(n, \mathbb{Q}) \to \mathrm{UT}(m+1, \mathbb{Q})$ *is an injective group homomorphism with admissible image.*

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Theorem

1. *The groups that admit free affine actions on $\mathbb{Z}^n$ for some n are precisely finitely generated torsion-free nilpotent groups.*
2. *Every locally residually torsion-free nilpotent group admits a free rigid affine action on a metric line.*
3. *Every free soluble group admits a free rigid affine action on a metric line.*

Recall (once more) that $BS(1, r)$ admits a free rigid action on $\mathbb{Z} \times \mathbb{R}$, via

$$\begin{array}{ll} a \cdot (m, x) & = (m, \quad x + 1) \\ t \cdot (m, x) & = (m + 1, \quad rx). \end{array}$$

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This can be naturally extended to an action on $\mathbb{R} \times \mathbb{R}$, and can be represented by admissible matrices via

$$a \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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So what other (non-nilpotent) groups of upper triangular matrices admit free affine actions on $\mathbb{R}^m$ for some m?

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Theorem

The group $T^*(n, \mathbb{R})$ admits an embedding in $T^*(n+m+1, \mathbb{R})$ with admissible image. Thus $T^*(n, \mathbb{R})$ admits a free rigid affine action on $\mathbb{R}^{n+m}$ (considered as an $\mathbb{R}^{n+m}$-tree).

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Then $B = U \rtimes D^*$, where U denotes unipotent matrices and D^* denotes diagonal matrices with positive diagonal entries.

Theorem

The group $T^*(n, \mathbb{R})$ admits an embedding in $T^*(n+m+1, \mathbb{R})$ with admissible image. Thus $T^*(n, \mathbb{R})$ admits a free rigid affine action on $\mathbb{R}^{n+m}$ (considered as an $\mathbb{R}^{n+m}$-tree).

The proof loosely follows an argument of John Milnor (see the proof of Theorem 1.2 in 'On Fundamental Groups of Complete Affinely Flat Manifolds' (Adv. Math. 1977)).

Example: $n = 3$

A typical element of $T^*(3, \mathbb{R})$ is expressible in the form ud where

$$u = \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \text{ and } d = \begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{pmatrix}.$$

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Now $\varphi(u) = \left(\begin{array}{ccc|c} 1 & -x/2 & y/2 & z - xy/2 \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$ so that

$$\varphi_0(u) = \begin{pmatrix} 1 & -x/2 & y/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } b(u) = \begin{pmatrix} z - xy/2 \\ y \\ x \end{pmatrix} \text{ where}$$

φ is our admissible embedding of $\mathrm{UT}(n, \mathbb{Q})$ in $\mathrm{UT}(m+1, \mathbb{Q})$.

Also $d^* = \begin{pmatrix} r/t & 0 & 0 \\ 0 & r/s & 0 \\ 0 & 0 & s/t \end{pmatrix}$ so that $\bar{\varphi}(ud) = \bar{\varphi}(u)\bar{\varphi}(d)$

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$$\bar{\varphi}(d) = \left(\begin{array}{ccc|ccc|c} r/t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r/s & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s/t & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & \log(r) \\ 0 & 0 & 0 & 0 & 1 & 0 & \log(s) \\ 0 & 0 & 0 & 0 & 0 & 1 & \log(t) \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

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More generally $\Lambda_1 \wr G$ admits such an action where G admits a free rigid affine action on a metric line.
Hence an iterated wreath product of ordered abelian groups admits such an action.

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1. G is right-orderable;
2. G admits a free order-preserving action on a linearly ordered set.

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Question: Suppose G has an order-preserving action on a linearly ordered set X. Under what conditions does X admit an equivariant embedding in a metric line $\hat{X}$ equipped with
(a) an affine action? (b) a rigid affine action?

We are working towards:

1. Let G be a group equipped with an order-preserving action on a linearly ordered set X. Then there exists an ordered abelian group Λ, an affine action of G on Λ and an equivariant embedding of X in Λ.

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1. Let G be a group equipped with an order-preserving action on a linearly ordered set X. Then there exists an ordered abelian group Λ, an affine action of G on Λ and an equivariant embedding of X in Λ.
2. Let G be a group equipped with a rigid order-preserving action on a linearly ordered set X. Then there exists an ordered abelian group Λ, a rigid affine action of G on Λ and an equivariant embedding of X in Λ.

Dziękuję bardzo!