Engel elements in groups of automorphisms of rooted trees

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Engel elements

Let G be a group and let $g \in G$.

- g is a *left Engel element* if for every $x \in G$
$$[x, g, \overset{n}{\ldots}, g] = 1 \quad \text{for some } n = n(g, x).$$
If n is independent of x, then g is a *bounded left Engel element*.
- $L(G)$ and $\bar{L}(G)$ denote the sets of left/bounded left Engel elements.
- g is a *right Engel element* if for every $x \in G$ there exists
$$[g, x, \overset{n}{\ldots}, x] = 1 \quad \text{for some } n = n(g, x).$$
If n is independent of x, then g is a *bounded right Engel element*.
- $R(G)$ and $\bar{R}(G)$ denote the sets of right/bounded right Engel elements.
- We have $R(G)^{-1} \subseteq L(G)$ and $\bar{R}(G)^{-1} \subseteq \bar{L}(G)$.

Engel groups and Burnside-like problems

- If $L(G) = G$ or, equivalently, $R(G) = G$, then G is an *Engel group*.
- Locally nilpotent groups are obviously Engel groups.

Is every Engel group locally nilpotent?

- Compare with the General Burnside Problem: is every periodic group locally finite?
- The Burnside Problem asks: is every group of finite exponent locally finite? We can raise a similar question for Engel groups.
- Let $n \in \mathbb{N}$. A group G is *n-Engel* if $[x, g, \overset{n}{\ldots}, g] = 1$ for all $x, g \in G$.

Is every n-Engel group locally nilpotent?

Positive answers

- Every Engel group G is locally nilpotent if:
  - G is finite (Zorn, 1936).
  - G is soluble (Gruenberg, 1953).
  - G satisfies the maximal condition (Baer, 1957).
  - G is linear (Garaščuk, Suprunenko, 1962).
  - G is a compact topological group (Medvedev, 2003).
- Every n-Engel group is locally nilpotent if $n \leq 4$. (Hopkins, 1929, $n = 2$; Heineken, 1961, $n = 3$; Havas, Vaughan-Lee, 2003, $n = 4$).
- Every residually finite n-Engel group is locally nilpotent. (Wilson, 1991)

Compare the latter with Zelmanov's positive solution of the Restricted Burnside Problem: every residually finite group of finite exponent is locally finite.

Negative answer: Golod groups

For every $d \geq 2$, there exists an infinite d-generator p-group G_d such that every subgroup of G_d with at most $d-1$ generators is nilpotent (finite).

- For $d \geq 3$, Golod groups are Engel, but not locally nilpotent.
- Golod groups also provide a negative answer to the General Burnside Problem.

It is still an open problem whether there exist n-Engel groups, for $n \geq 5$, that are not locally nilpotent.

Distinguished subgroups inside $L(G)$, $\bar{L}(G)$, $R(G)$, $\bar{R}(G)$

For an arbitrary group G:

- $L(G)$ contains the Hirsch-Plotkin radical of G (the unique maximal locally nilpotent normal subgroup of G).
- If $\langle g \rangle$ is subnormal in G then $g \in \bar{L}(G)$.
- So $\bar{L}(G)$ contains the Baer radical of G (the unique maximal normal subgroup of G in which all cyclic subgroups are subnormal).
- $R(G)$ contains the hypercentre.
- $\bar{R}(G)$ contains the ω-centre, that is, $\cup_{i \in \mathbb{N}} Z_i(G)$.

From Robinson's book *A course in the theory of groups*:

> The major goal of Engel theory is to find conditions which will guarantee that $L(G)$, $\bar{L}(G)$, $R(G)$, and $\bar{R}(G)$ are subgroups which coincide with the Hirsch-Plotkin radical, the Baer radical, the hypercenter and the ω-center respectively.

Are the sets $L(G)$, $\bar{L}(G)$, $R(G)$, $\bar{R}(G)$ subgroups of G?

Do $L(G)$, $\bar{L}(G)$, $R(G)$ and $\bar{R}(G)$ coincide with the Hirsch-Plotkin radical, the Baer radical, the hypercentre and the ω-centre?

- True if every abelian subgroup of G is finitely generated.
- If G is soluble, it is true for $L(G)$ and $\bar{L}(G)$, but it may fail for $R(G)$ and $\bar{R}(G)$.
- It fails in every case in Golod's groups.

In all cases above where the answer is negative, the corresponding Engel sets are nevertheless subgroups.

Perhaps they are always subgroups?

Grigorchuk group comes into play

Let Γ be the first Grigorchuk group.

Bludov, 2005 (unpublished)

The wreath product $\Gamma \wr D_8$, with the natural action of D_8 on 4 points, can be generated by left Engel elements but it is not an Engel group.

Bartholdi, 2016

$L(\Gamma)$ coincides with the set of elements of order 2 in Γ. So Γ can be generated by left Engel elements but it is not an Engel group.

Thus in both cases $L(G)$ is not a subgroup.

Can the Grigorchuk group, or similar groups, give examples where $\bar{L}(G)$, $R(G)$ or $\bar{R}(G)$ are not subgroups?

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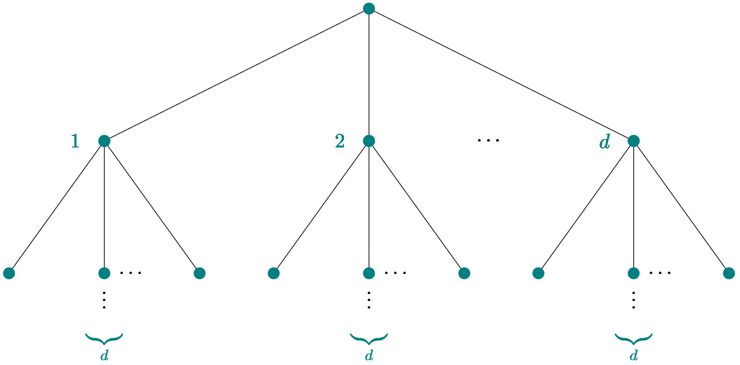
Spherically homogeneous rooted trees

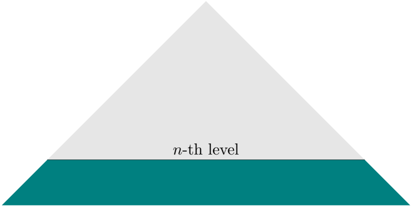


- The root is a distinguished vertex and the tree is infinite.
- Spherically homogeneous: the number of descendants is the same at every level.
- But it can be different at different levels.
- If it is always the same, say d, we call it the d-adic tree. (Picture)

Automorphism group of $\mathcal{T}$

- Automorphisms of $\mathcal{T}$: bijections of the vertices that preserve incidence.
- $\operatorname{Aut}\mathcal{T}$ is a group with respect to composition.
- It is the inverse limit of the finite groups given by the automorphism groups of the truncated trees at every level.
- So $\operatorname{Aut}\mathcal{T}$ is a *profinite group*.

Level stabilizers

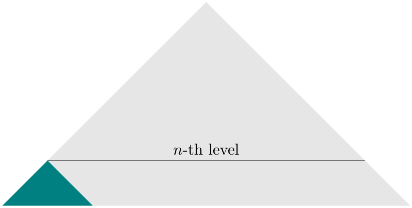


- Let L_n be the set of vertices on the nth level of the tree.
- The nth level stabilizer $\mathrm{st}(n)$ is the pointwise stabilizer of L_n.
- If $\mathcal{T}_n$ is the subtree hanging from any vertex at level n:

$$\mathrm{st}(n) \cong \operatorname{Aut}\mathcal{T}_n \times \overset{|L_n|}{\cdots} \times \operatorname{Aut}\mathcal{T}_n.$$

  If $\mathcal{T}$ is d-adic, $\mathcal{T}_n \cong \mathcal{T}$, which shows the self-similar structure of $\operatorname{Aut}\mathcal{T}$.
- If $G \leq \operatorname{Aut}\mathcal{T}$, we set $\mathrm{st}_G(n) = G \cap \mathrm{st}(n)$.

Rigid stabilizers



- The rigid vertex stabilizer of a vertex v, $\mathrm{rst}(v)$, is the subgroup of all automorphisms fixing all vertices outside the subtree hanging from v.
- $\mathrm{st}(n) = \prod_{v \in L_n} \mathrm{rst}(v)$.
- If $G \leq \operatorname{Aut} \mathcal{T}$, we set $\mathrm{rst}_G(v) = G \cap \mathrm{rst}(v)$.
- It is not generally the case that $\mathrm{st}_G(n) = \prod_{v \in L_n} \mathrm{rst}_G(v)$.
- Let $\mathrm{rst}_G(n) = \prod_{v \in L_n} \mathrm{rst}_G(v)$, the *rigid level stabilizer* of G at level n.
- It is the largest "geometrical" direct product inside $\mathrm{st}_G(n)$.

Describing automorphisms of $\mathcal{T}$

- The simplest type are *rooted automorphisms*: given $\sigma \in S_{d_1}$, they simply permute the d_1 subtrees hanging from the root according to σ.
- Rooted automorphisms form a subgroup $R(\mathcal{T})$ isomorphic to S_{d_1}.
- We have $\operatorname{Aut}\mathcal{T} = \operatorname{st}(1) \rtimes R(\mathcal{T})$.
- If we identify $\operatorname{st}(1)$ with $\operatorname{Aut}\mathcal{T}_1 \times \overset{d_1}{\cdots} \times \operatorname{Aut}\mathcal{T}_1$, every $f \in \operatorname{Aut}\mathcal{T}$ can be written as

$$f = (f_1, \dots, f_{d_1})a,$$

  where $f_i \in \operatorname{Aut}\mathcal{T}_1$ and a is rooted.
- This can be used to define automorphisms, and the definition can be recursive.
- If $\mathcal{T}$ is the binary tree and a is rooted corresponding to $(1\ 2)$, let

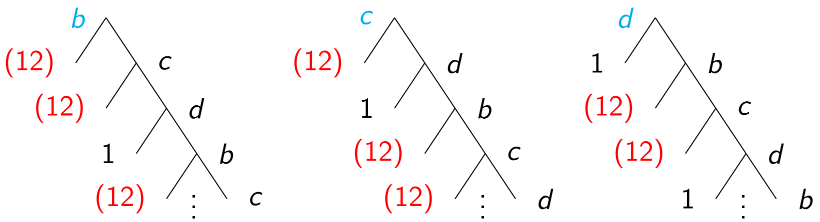
$$b = (1, b)a.$$

  How does b act on $\mathcal{T}$?

The first Grigorchuk group

$$\Gamma = \langle a, b, c, d \rangle,$$

where a is rooted corresponding to $(1\ 2)$, $b = (a, c)$, $c = (a, d)$, $d = (1, b)$.



Γ gives a negative answer to the General Burnside Problem, and it has many other interesting properties. Remember $L(\Gamma)$ is not a subgroup!

GGS-groups

If $\mathcal{T}$ is a d-adic tree, then a *GGS-group* (Grigorchuk-Gupta-Sidki group) is given by two generators:

- A rooted automorphism a corresponding to $(1\ 2\ \ldots\ d)$.
- A recursively defined automorphism b via

$$b = (a^{e_1}, \ldots, a^{e_{d-1}}, b)$$

for some tuple $\mathbf{e} = (e_1, \ldots, e_{d-1})$.

- If d is a prime this group is a negative solution to the General Burnside Problem if and only if

$$e_1 + \cdots + e_{d-1} \equiv 0 \mod d.$$

- The case of the vector $\mathbf{e} = (1, -1, 0, \ldots, 0)$ is the famous Gupta-Sidki group.

(Weakly) branch groups

Let G be a subgroup of $\operatorname{Aut} \mathcal{T}$ acting transitively on every level L_n.

- G is *branch* if $|G : \operatorname{rst}_G(n)| < \infty$ for all n.
- G is *weakly branch* if $\operatorname{rst}_G(n) \neq 1$ for all n.

- These groups try to approximate the behaviour of the full group $\operatorname{Aut} \mathcal{T}$, where $\operatorname{rst}(n) = \operatorname{st}(n)$ is as large as possible.
- The most important families of subgroups of $\operatorname{Aut} \mathcal{T}$ consist almost entirely of branch groups, and if not branch, the groups are usually weakly branch.
- The first Grigorchuk group is branch, and also the Gupta-Sidki p-groups.
- In the infinite family of the GGS-groups, all groups are branch, with only one exception ($\mathbf{e}$ constant), which is weakly branch.

Finitary automorphisms

- Every automorphism $f \in \operatorname{Aut} \mathcal{T}$ can be described by giving, for every vertex v, the permutation $f_{(v)}$ that says how f sends the descendants of v to the descendants of $f(v)$.
- $f_{(v)}$ is called the *label* of f at v.
- In a rooted automorphism, the labels are trivial at all vertices other than the root.
- We say that f is *finitary* if its labels are trivial at all but finitely many vertices of the tree.
- Finitary automorphisms are of finite order and they form a locally finite subgroup $\mathcal{F}$ of $\operatorname{Aut} \mathcal{T}$.
- Let $\mathcal{T}$ be the p-adic tree for a prime p, and fix a p-cycle σ in S_p. Then the finitary automorphisms whose labels are all powers of σ form a locally finite p-group $\mathcal{F}_p$.
- Both $\mathcal{F}$ and $\mathcal{F}_p$ are branch groups, since they satisfy $\operatorname{rst}_G(n) = \operatorname{st}_G(n)$ for all $n \geq 1$.

Fractal groups

Assume $\mathcal{T}$ is a d-adic tree. Then:

- The subtree $\mathcal{T}_v$ hanging from every vertex v is isomorphic to $\mathcal{T}$.
- If $f \in \operatorname{Aut}\mathcal{T}$ fixes v, then it induces by restriction an automorphism f_v of $\mathcal{T}_v$, so also of $\mathcal{T}$.

Let G be a subgroup of $\operatorname{Aut}\mathcal{T}$, where $\mathcal{T}$ is a d-adic tree. We say that G is *fractal* if for every vertex v, the set

$$\{f_v \mid f \in G \text{ and fixes } v\}$$

is equal to G.

- Obviously, $\operatorname{Aut}\mathcal{T}$ is fractal.
- The first Grigorchuk group Γ is fractal.
- All GGS-groups are fractal.

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A search inside Aut $\mathcal{T}$

Taking into account that:

- There is a similarity between Engel problems and Burnside-type problems.
- Many negative solutions to the General Burnside Problem are (weakly) branch/fractal subgroups of Aut $\mathcal{T}$.
- $L(\Gamma) = \{x \in \Gamma \mid x^2 = 1\}$ is not a subgroup.

It is natural to search inside Aut $\mathcal{T}$ for groups where $L(G)$, $\bar{L}(G)$, $R(G)$ or $\bar{R}(G)$ are not subgroups.

First negative results

Bartholdi, 2016

In the Gupta-Sidki 3-group $L(G) = 1$. So also $\bar{L}(G) = R(G) = \bar{R}(G) = 1$.

Noce, Tortora, 2018

$\bar{L}(\Gamma) = R(\Gamma) = \bar{R}(\Gamma) = 1$.

$L(G)$ in fractal groups

Let $\mathcal{T}$ be a p-adic tree, where p is a prime.

- A Sylow pro-p subgroup $\mathcal{P}$ of $\operatorname{Aut}\mathcal{T}$ is any inverse limit of Sylow p-subgroups of the automorphism groups of the truncated trees.
- For example, if we fix a p-cycle σ, then $\mathcal{P}$ can be chosen as the set of automorphisms all of whose labels are powers of σ.
- The Grigorchuk group lies in a Sylow pro-2 subgroup of the binary tree.
- All GGS-groups lie in a Sylow pro-p subgroup of the p-adic tree.

F-A, Garreta, Noce, 2018

Let G be a fractal group lying in a Sylow pro-p subgroup of $\operatorname{Aut}\mathcal{T}$. Then:

- If $|G' : \mathrm{st}_G(1)'| = \infty$ then $L(G) = 1$.
- If G is non-abelian and has torsion-free abelianization then $L(G) = 1$.

Application to specific fractal groups

In the following groups one has $L(G) = \bar{L}(G) = R(G) = \bar{R}(G) = 1$:

- The Basilica group, defined on the binary tree by

$$a = (1, b) \quad \text{and} \quad b = (1, a)\,(1\ 2).$$

- The Brunner-Sidki-Vieira group, defined on the binary tree by

$$a = (1, a^{-1})\,(1\ 2) \quad \text{and} \quad b = (1, b)\,(1\ 2).$$

- The GGS-group with constant defining vector.

Engel sets in weakly branch groups

F-A, Noce, Tracey, 2019

Let G be a weakly branch group. Then:

- $\bar{L}(G) = 1$.
- If $L(G)$ contains non-trivial elements of finite order then:
  - All such elements have p-power order for some prime p.
  - $\mathrm{rst}_G(n)$ is a p-group for some $n \geq 1$.

F-A, Noce, Tracey, 2019

Let G be a weakly branch group. If $\mathrm{rst}_G(n)$ is not Engel for any n, then $R(G) = 1$.

If $\mathcal{T}$ is a p-adic tree and $\mathcal{F}_p$ is the group of finitary automorphisms of a Sylow pro-p subgroup $\mathcal{P}$ of $\operatorname{Aut}\mathcal{T}$, then:

- $\mathcal{F}_p$ is branch.
- $\mathcal{F}_p$ is Engel, so $R(\mathcal{F}_p) = \mathcal{F}_p \neq 1$.

$L(G)$ for branch groups

F-A, Noce, Tracey, 2019

Let G be a branch group. Then

- If G is not periodic, then $L(G) = 1$.
- If G is periodic, then $L(G)$ consists of p-elements for some prime p.
- If $L(G) \neq 1$, then G is virtually a p-group for the same prime as in (ii).

Compare (ii) and (iii) with the situation in the Grigorchuk group:

- $L(\Gamma)$ consists of all elements of order 2 in Γ.
- Γ is a 2-group.

Observe that he prime p can be arbitrary in (ii) and (iii):

- $\mathcal{F}_p$ is a branch p-group.
- $\mathcal{F}_p$ is Engel, so $L(\mathcal{F}_p) = \mathcal{F}_p$.
- But contrary to Γ, $\mathcal{F}_p$ is not finitely generated.

Some applications

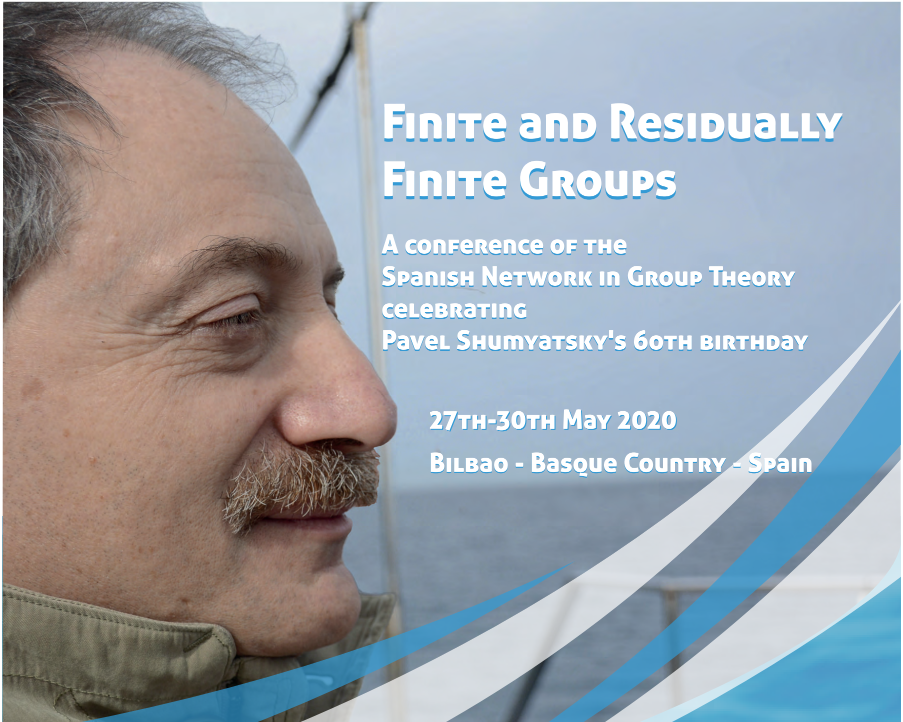
- $L(\mathrm{Aut}\,\mathcal{T}) = 1$.
- $L(\mathcal{F}) = 1$, where $\mathcal{F}$ is the group of finitary automorphisms of $\mathcal{T}$.
- All GGS-groups satisfy $\overline{L}(G) = 1$ and, if they are not periodic, $L(G) = 1$.
- The Hanoi tower group satisfies $L(G) = 1$. In the case of 3 pegs, it acts on the ternary tree and is generated by

$$a = (1, 1, a)\,(1\ 2), \quad b = (1, b, 1)\,(1\ 3), \quad c = (c, 1, 1)\,(2\ 3).$$

Similarly for an arbitrary finite number of pegs.

Open questions

- Can a weakly branch group contain Engel elements of infinite order?
- Is $R(G) = 1$ for every *finitely generated* weakly branch group?
- Are there any *finitely generated* branch groups for which the set $L(G)$ consists of p-elements for an *odd* prime p?
- Can a *finitely generated* branch group be Engel? Observe that:
  - Weakly branch groups cannot satisfy a law, so they cannot be n-Engel for any n.
  - However, as $\mathcal{F}_p$ shows, weakly branch groups can be Engel. But $\mathcal{F}_p$ is not finitely generated.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