Some anti-geometric groups

Sam Corson
(joint with Saharon Shelah)

Groups and their actions
9 September 2019

(In)actions

Groups can be interesting because of their actions.

(In)actions

Groups can be interesting because of their actions.

For example: hyperbolic groups, acylindrically hyperbolic groups.

(In)actions

Groups can be interesting because of their actions.

For example: hyperbolic groups, acylindrically hyperbolic groups.

Groups can also be interesting because of limitations of actions.

(In)actions

Groups can be interesting because of their actions.

For example: hyperbolic groups, acylindrically hyperbolic groups.

Groups can also be interesting because of limitations of actions.

For example: property FA, property FH, amenability.

Strongly bounded groups

Goal: Create groups which are interesting because of limitations on their isometric actions.

Strongly bounded groups

Goal: Create groups which are interesting because of limitations on their isometric actions.

Definition

A group G is *strongly bounded* if every abstract action of G by isometries on a metric space has bounded orbits [2].

Strongly bounded groups

Goal: Create groups which are interesting because of limitations on their isometric actions.

Definition

A group G is *strongly bounded* if every abstract action of G by isometries on a metric space has bounded orbits [2].

Example

Finite groups

Some necessary conditions

Definition

G is not ω-*cofinal* if there is not a chain of proper subgroups $G_0 < G_1 < \cdots$ for which $G = \bigcup_{n\in\omega} G_n$.

Some necessary conditions

Definition

G is not ω-*cofinal* if there is not a chain of proper subgroups $G_0 < G_1 < \cdots$ for which $G = \bigcup_{n\in\omega} G_n$.

Definition

G is *Cayley bounded* if for every generating set X the Cayley graph $\Gamma(G, X)$ is of bounded diameter.

Some necessary conditions

Definition

G is not ω-*cofinal* if there is not a chain of proper subgroups $G_0 < G_1 < \cdots$ for which $G = \bigcup_{n\in\omega} G_n$.

Definition

G is *Cayley bounded* if for every generating set X the Cayley graph $\Gamma(G, X)$ is of bounded diameter.

The conjunction of these two properties is sufficient for a group to be strongly bounded. Strongly bounded groups cannot be countably infinite [2].

Some strongly bounded groups

A few examples of such groups:

Some strongly bounded groups

A few examples of such groups:

- (Shelah 1980) Weird groups arising as examples of Jónsson groups ($\lambda^+ = 2^\lambda$).

Some strongly bounded groups

A few examples of such groups:

- (Shelah 1980) Weird groups arising as examples of Jónsson groups ($\lambda^+ = 2^\lambda$).
- (Bergman 2006) The full symmetric group S_X on a set X.

Some strongly bounded groups

A few examples of such groups:

- (Shelah 1980) Weird groups arising as examples of Jónsson groups ($\lambda^+ = 2^\lambda$).
- (Bergman 2006) The full symmetric group S_X on a set X.
- (Droste, Göbel 2005) Self homeomorphisms of the Cantor set, of $\mathbb{Q}$, of $\mathbb{R}$.

Some strongly bounded groups

A few examples of such groups:

- (Shelah 1980) Weird groups arising as examples of Jónsson groups ($\lambda^+ = 2^\lambda$).
- (Bergman 2006) The full symmetric group S_X on a set X.
- (Droste, Göbel 2005) Self homeomorphisms of the Cantor set, of $\mathbb{Q}$, of $\mathbb{R}$.
- (de Cornulier 2006) $\prod_I H$ with H a finite perfect group; ω_1-existentially closed groups.

What cardinalities can arise?

The above groups have the following in common: they are finite or are of cardinality $\kappa^{\aleph_0}$ for some cardinal κ (often $\kappa = 2$).

What cardinalities can arise?

The above groups have the following in common: they are finite or are of cardinality $\kappa^{\aleph_0}$ for some cardinal κ (often $\kappa = 2$).

Particularly the infinite examples have cardinality $\geq 2^{\aleph_0}$.

What cardinalities can arise?

The above groups have the following in common: they are finite or are of cardinality $\kappa^{\aleph_0}$ for some cardinal κ (often $\kappa = 2$).

Particularly the infinite examples have cardinality $\geq 2^{\aleph_0}$.

Question

[2] Can a strongly bounded group have cardinality $\aleph_1$ (without assuming $\aleph_1 = 2^{\aleph_0}$)?

YES

Theorem

(C., Shelah 2019) Let λ be a cardinal of uncountable cofinality and K be a group such that $|K| < \lambda$. Then there exists a strongly bounded group $G \geq K$ which is of cardinality λ,

YES

Theorem

(C., Shelah 2019) Let λ be a cardinal of uncountable cofinality and K be a group such that $|K| < \lambda$. Then there exists a strongly bounded group $G \geq K$ which is of cardinality λ, except possibly when $\lambda = \mu^+$ where $\mathrm{cof}(\mu) = \omega$ and μ is a limit of weakly inaccessible cardinals.

YES

Theorem

(C., Shelah 2019) Let λ be a cardinal of uncountable cofinality and K be a group such that $|K| < \lambda$. Then there exists a strongly bounded group $G \geq K$ which is of cardinality λ, except possibly when $\lambda = \mu^+$ where $\mathrm{cof}(\mu) = \omega$ and μ is a limit of weakly inaccessible cardinals.

For example we can let $\kappa = \aleph_1, \aleph_2, \aleph_3, \aleph_{\omega+1}, \aleph_{\omega+2}, \aleph_{\omega_1}$, etc.

YES

Theorem

(C., Shelah 2019) Let λ be a cardinal of uncountable cofinality and K be a group such that $|K| < \lambda$. Then there exists a strongly bounded group $G \geq K$ which is of cardinality λ, except possibly when $\lambda = \mu^+$ where $\mathrm{cof}(\mu) = \omega$ and μ is a limit of weakly inaccessible cardinals.

For example we can let $\kappa = \aleph_1, \aleph_2, \aleph_3, \aleph_{\omega+1}, \aleph_{\omega+2}, \aleph_{\omega_1}$, etc. The uncountable cofinality condition cannot be dropped.

Sketch of proof when $\lambda = \aleph_1$

Lemma

For each $n \geq 1$ *there is a group word* $w(x_0, x_1, \ldots, x_{n-1}, y)$ *such that the following holds: If* G *is a group and* $f : (G \setminus \{1_G\})^n \to G$ *then there exist group* H *and* $c \in H$ *such that*

(a) $G \leq H$;

(b) $c \in H \setminus G$;

(c) *for all* $\overline{g} \in (G \setminus \{1_G\})^n$ *we have* $w(\overline{g}, c) = f(\overline{g})$;

(d) $H = \langle G \cup \{c\} \rangle$.

Sketch of proof when $\lambda = \aleph_1$

Definition

If X is a set and $n \in \omega$ we let $[X]^n$ denote the set of subsets of X of cardinality n.

Sketch of proof when $\lambda = \aleph_1$

Definition

If X is a set and $n \in \omega$ we let $[X]^n$ denote the set of subsets of X of cardinality n.

Conventions

An ordinal is the set of ordinals which are strictly below it ($0 = \emptyset$, $1 = \{0\}$, $\omega + 1 = \{0, 1, \ldots, \omega\}$). A cardinal is the least ordinal of its cardinality.

Sketch of proof when $\lambda = \aleph_1$

Definition

If X is a set and $n \in \omega$ we let $[X]^n$ denote the set of subsets of X of cardinality n.

Conventions

An ordinal is the set of ordinals which are strictly below it ($0 = \emptyset$, $1 = \{0\}$, $\omega + 1 = \{0, 1, \ldots, \omega\}$). A cardinal is the least ordinal of its cardinality.

Theorem

(Todorčević 1987) There exists a function $f : [\aleph_1]^2 \to \aleph_1$ such that if $Y \subseteq \aleph_1$ is uncountable then $f([Y]^2) = \aleph_1$.

Sketch of proof when $\lambda = \aleph_1$

Setup

Sketch of proof when $\lambda = \aleph_1$

Setup

- $\{\beta_\alpha\}_{\alpha<\aleph_1}$ = the set of limit ordinals less than $\aleph_1$, ordered appropriately

Sketch of proof when $\lambda = \aleph_1$

Setup

- $\{\beta_\alpha\}_{\alpha<\aleph_1}$ = the set of limit ordinals less than $\aleph_1$, ordered appropriately
- $f : [\aleph_1]^2 \to \aleph_1$ as in Todorčević's theorem

Sketch of proof when $\lambda = \aleph_1$

Setup

- $\{\beta_\alpha\}_{\alpha<\aleph_1}$ = the set of limit ordinals less than $\aleph_1$, ordered appropriately
- $f : [\aleph_1]^2 \to \aleph_1$ as in Todorčević's theorem
- Without loss of generality $f([\beta_\alpha]^2) \subseteq \beta_\alpha$

Sketch of proof when $\lambda = \aleph_1$

Construction

Inductively endow each β_α with a group structure G_α.

Sketch of proof when $\lambda = \aleph_1$

Construction

Inductively endow each β_α with a group structure G_α.
Let β_0 be given a group structure such that $K \leq G_0$.

Sketch of proof when $\lambda = \aleph_1$

Construction

Inductively endow each β_α with a group structure G_α.
Let β_0 be given a group structure such that $K \leq G_0$.
(Successor) If β_α has group structure G_α then give $\beta_{\alpha+1}$ a group structure such that

Sketch of proof when $\lambda = \aleph_1$

Construction

Inductively endow each β_α with a group structure G_α.
Let β_0 be given a group structure such that $K \leq G_0$.
(Successor) If β_α has group structure G_α then give $\beta_{\alpha+1}$ a group structure such that

- $G_{\alpha+1} = \langle \{\beta_\alpha\} \cup G_\alpha \rangle$

Sketch of proof when $\lambda = \aleph_1$

Construction

Inductively endow each β_α with a group structure G_α.
Let β_0 be given a group structure such that $K \leq G_0$.
(Successor) If β_α has group structure G_α then give $\beta_{\alpha+1}$ a group structure such that

- $G_{\alpha+1} = \langle \{\beta_\alpha\} \cup G_\alpha \rangle$
- $w(\overline{g}, \beta_\alpha) = f(\overline{g})$ when $\overline{g} = (g_0, g_1)$.

Sketch of proof when $\lambda = \aleph_1$

Construction

Inductively endow each β_α with a group structure G_α.
Let β_0 be given a group structure such that $K \leq G_0$.
(Successor) If β_α has group structure G_α then give $\beta_{\alpha+1}$ a group structure such that

- $G_{\alpha+1} = \langle \{\beta_\alpha\} \cup G_\alpha \rangle$
- $w(\overline{g}, \beta_\alpha) = f(\overline{g})$ when $\overline{g} = (g_0, g_1)$.

(Limit) If β_α has a group structure G_α for all $\alpha < \gamma$ then give β_γ the group structure $G_\gamma = \bigcup_{\alpha<\gamma} G_\alpha$.

Sketch of proof when $\lambda = \aleph_1$

Does it work?

Sketch of proof when $\lambda = \aleph_1$

Does it work?

Suppose that $Y_0 \subseteq Y_1 \subseteq \cdots$ with

- $Y_{m+1} \supseteq Y_m \cup Y_m^{-1} \cup Y_m Y_m$

Sketch of proof when $\lambda = \aleph_1$

Does it work?

Suppose that $Y_0 \subseteq Y_1 \subseteq \cdots$ with

- $Y_{m+1} \supseteq Y_m \cup Y_m^{-1} \cup Y_m Y_m$
- $\bigcup_{m\in\omega} Y_m = G$

Sketch of proof when $\lambda = \aleph_1$

Does it work?

Suppose that $Y_0 \subseteq Y_1 \subseteq \cdots$ with

- $Y_{m+1} \supseteq Y_m \cup Y_m^{-1} \cup Y_m Y_m$
- $\bigcup_{m \in \omega} Y_m = G$

We show $G = Y_M$ for some M.

Sketch of proof when $\lambda = \aleph_1$

Does it work?

Suppose that $Y_0 \subseteq Y_1 \subseteq \cdots$ with

- $Y_{m+1} \supseteq Y_m \cup Y_m^{-1} \cup Y_m Y_m$
- $\bigcup_{m\in\omega} Y_m = G$

We show $G = Y_M$ for some M.
Certainly $Y_{M_0} \cap \{\beta_\alpha\}_{\alpha<\aleph_1}$ is uncountable for some M_0.

Sketch of proof when $\lambda = \aleph_1$

Does it work?

Suppose that $Y_0 \subseteq Y_1 \subseteq \cdots$ with

- $Y_{m+1} \supseteq Y_m \cup Y_m^{-1} \cup Y_m Y_m$
- $\bigcup_{m \in \omega} Y_m = G$

We show $G = Y_M$ for some M.
Certainly $Y_{M_0} \cap \{\beta_\alpha\}_{\alpha < \aleph_1}$ is uncountable for some M_0. Given any $\gamma \in G$ we select $\beta_{\alpha_0}, \beta_{\alpha_1} \in Y_{M_0}$ for which $f(\{\beta_{\alpha_0}, \beta_{\alpha_1}\}) = \gamma$.

Sketch of proof when $\lambda = \aleph_1$

Does it work?

Suppose that $Y_0 \subseteq Y_1 \subseteq \cdots$ with

- $Y_{m+1} \supseteq Y_m \cup Y_m^{-1} \cup Y_m Y_m$
- $\bigcup_{m\in\omega} Y_m = G$

We show $G = Y_M$ for some M.
Certainly $Y_{M_0} \cap \{\beta_\alpha\}_{\alpha<\aleph_1}$ is uncountable for some M_0. Given any $\gamma \in G$ we select $\beta_{\alpha_0}, \beta_{\alpha_1} \in Y_{M_0}$ for which $f(\{\beta_{\alpha_0}, \beta_{\alpha_1}\}) = \gamma$.Select $\beta_{\alpha_2} \in Y_{M_0}$ for which $\beta_{\alpha_2} > \beta_{\alpha_0}, \beta_{\alpha_1}$.

Sketch of proof when $\lambda = \aleph_1$

Does it work?

Suppose that $Y_0 \subseteq Y_1 \subseteq \cdots$ with

- $Y_{m+1} \supseteq Y_m \cup Y_m^{-1} \cup Y_m Y_m$
- $\bigcup_{m\in\omega} Y_m = G$

We show $G = Y_M$ for some M.
Certainly $Y_{M_0} \cap \{\beta_\alpha\}_{\alpha<\aleph_1}$ is uncountable for some M_0. Given any $\gamma \in G$ we select $\beta_{\alpha_0}, \beta_{\alpha_1} \in Y_{M_0}$ for which $f(\{\beta_{\alpha_0}, \beta_{\alpha_1}\}) = \gamma$.Select $\beta_{\alpha_2} \in Y_{M_0}$ for which $\beta_{\alpha_2} > \beta_{\alpha_0}, \beta_{\alpha_1}$.
Now $w(\beta_0, \beta_1, \beta_2) = \gamma$, so $\gamma \in Y_{M_0+length(w)}$.

Locally finite?

Theorem

(C., Shelah 2019) Suppose that there exists an increasing sequence $\{Y_\alpha\}_{\alpha<\aleph_1}$ of sets of Lebesgue measure zero such that every set of measure zero is eventually included in the sequence. Then for every nontrivial finite perfect group H there is a strongly bounded $G \leq \prod_\omega H$ of cardinality $\aleph_1$.

Locally finite?

Theorem

(C., Shelah 2019) Suppose that there exists an increasing sequence $\{Y_\alpha\}_{\alpha<\aleph_1}$ *of sets of Lebesgue measure zero such that every set of measure zero is eventually included in the sequence. Then for every nontrivial finite perfect group* H *there is a strongly bounded* $G \leq \prod_\omega H$ *of cardinality* $\aleph_1$*.*

For any cardinal κ of uncountable cofinality there exists a model of ZFC in which the hypothesis is satisfied and $\kappa = 2^{\aleph_0}$.

Thank you.

G. Bergman, *Generating infinite symmetric groups*, Bull. London Math. Soc. 38 (2006), 429-440.

Y. de Cornulier, *Strongly bounded groups and infinite powers of finite groups*, Comm. Alg. 34 (2006), 2337-2345.

M. Droste, R. Göbel, *Uncountable cofinalities of permutation groups*, J. London Math. Soc. 71 (2005), 335-344.

S. Shelah, *On a problem of Kurosh, Jónsson groups, and applications*, Word Problems II. North-Holland Publ. Company (1980), 373-394.