Equations in acylindrically hyperbolic groups.

Algebraic, verbal, and existential closedness of subgroups of groups

Oleg Bogopolski

Groups and their actions, Gliwice, 9-13 September 2019

Papers

1. O. Bogopolski, A periodicity theorem for acylindrically hyperbolic groups, 2018. https://arxiv.org/pdf/1805.05941.pdf

2. O. Bogopolski, Equations in acylindrically hyperbolic groups and verbal closedness, 2018. https://arxiv.org/pdf/1805.08071.pdf

3. O. Bogopolski, On finite systems of equations in acylindrically hyperbolic groups, 2019. https://arxiv.org/pdf/1903.10906.pdf

Acylindric actions on metric spaces

Def. (Bowditch, Osin) An isometric action of a group G on a metric space S is called acylindrical if for every $\varepsilon > 0$ there exist $R, N > 0$ such that for every two points $x, y \in S$ with $d(x, y) \geqslant R$ we have

$$\left|\{g \in G \mid d(x, gx) \leqslant \varepsilon \text{ and } d(y, gy) \leqslant \varepsilon\}\right| \leqslant N.$$

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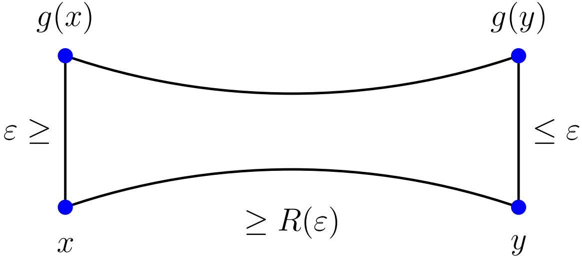

Acylindrically hyperbolic groups

Def. (Osin) A group G is called acylindrically hyperbolic if it satisfies one of the following equivalent conditions:

(AH_1) There exists a generating set X of G such that the corresponding Cayley graph $\Gamma(G, X)$ is hyperbolic, $|\partial\Gamma(G, X)| > 2$, and the natural action of G on $\Gamma(G, X)$ is acylindrical.

(AH_2) G admits a non-elementary acylindrical action on a hyperbolic space.

In the case (AH_1), we also say that G is acylindrically hyperbolic with respect to X.

Examples of acylindrically hyperbolic groups

1. Every non-(virtually cyclic) relatively hyperbolic group with proper peripheral subgroups.
2. $\mathrm{MCG}(\Sigma_{g,p})$ except $g = 0, p \leqslant 3$.
3. $\mathrm{Out}(F_n)$, $n \geqslant 2$.
4. Noncyclic directly indecomposable RAAG's.
5. For every compact orientable irreducible 3-manifold M, the fundamental group $\pi_1(M)$ is either
   - virtually polycyclic, or
   - acylindrically hyperbolic, or
   - M is Seifert fibert.

   In the latter case $\pi_1(M)$ contains a normal subgroup $N \cong \mathbb{Z}$ such that $\pi_1(M)/N$ is acylindrically hyperbolic.

Two interesting theorems

Thm. (Osin) Groups of deficiency at least 2 are acylindrically hyperbolic.

Thm. (Minasyan, Osin) Let G be a group acting minimally on a simplicial tree. Suppose that G does not fix any point on ∂T and there exist two vertices $u, v \in T$ such that the pointwise stabilizer of $\{u, v\}$ is finite (i.e. $|St(u) \cap St(v)| < \infty$). Then G is either virtually cyclic or acylindrically hyperbolic.

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Cor. (Minasyan, Osin)

a) Let $G = A *_C B$, where $A \neq C \neq B$ and C is weakly malnormal, i.e. there exists $g \in G$ with $|C^g \cap C| < \infty$. Then G is either virtually cyclic or acylindrically hyperbolic.

b) Let $G = \langle A, t \mid t^{-1}Ct = D \rangle$, where $C \neq A \neq D$ and C is weakly malnormal. Then G is either virtually cyclic or acylindrically hyperbolic.

Clean groups etc.

Most theorems will be given for clean groups.

Def.

• A group H is called clean if H does not have nontrivial finite normal subgroups.

• A group G is finitely generated over a subgroup H if there exists a finite subset $A \subset G$ such that $G = \langle A, H \rangle$.

Equations with constants from a group H

Let $X = \{x_1, x_2, \dots\}$ be a countably infinite set of variables, and let H be a group. An equation with variables $x_1, \dots, x_n$ and constants from H is an arbitrary expression

$$f(x_1, \dots, x_n; H) = 1,$$

where the left side is a word in the alphabet $\{x_1, \dots, x_n\}^{\pm} \cup H$.

In other words $f(x_1, \dots, x_n; H)$ lies in the free product $F(X) * H$, where $F(X)$ is the free group with basis X.

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For any overgroup G of H, the set of solutions of f in G is denoted by

$$V_G(f).$$

Equationally Noetherian groups

Def. A group H is called equationally Noetherian if every system of equations with constants from H and a finite number of variables is equivalent to a finite subsystem.

Examples of equationally noetherian groups

(1) All linear groups over a commutative noetherian unitary ring. (over a field – Bryant; extended by Baumslag, Myasnikov and Remeslennikov).

(2) Finitely generated abelian-by-nilpotent group (Bryant)

(3) Rigid solvable groups (Romanovskii and Gupta).
[A solvable group G is called *rigid* if it possesses a normal series of the form $G = G_1 > G_2 > \cdots > G_m > G_{m+1} = 1$, where the quotients G_i/G_{i+1} are abelian and torsion-free as right $\mathbb{Z}[G/G_i]$-modules. In particular, free solvable groups are rigid.]

(4) If A and B are equationally noetherian groups, then their free product $A * B$ is also equationally noetherian (Sela).

(5) Hyperbolic groups are equationally noetherian (torsion-free case – Sela; extended by Reinfeldt and Weidmann).

(6) Suppose that G is a relatively hyperbolic group with respect to a finite collection of subgroups $\{H_1, \ldots, H_n\}$. Then G is equationally noetherian if and only if each H_i is equationally noetherian (Groves and Hull).

Systems of equations $\to$ a single equation

Thm A. (Bogo, 2018) Let H be a clean acylindrically hyperbolic group and let

$$S : \begin{cases} s_1(x_1, \ldots, x_n; H) = 1, \\ \ldots \\ s_k(x_1, \ldots, x_n; H) = 1, \end{cases}$$

be a finite system of equations with constants from H. Then there exists a single equation $f(x_1, \ldots, x_n; H) = 1$ with $V_H(S) = V_H(f)$.

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Cor A1. (Bogo, 2018) Let H be a clean nonelementary hyperbolic group and let $S \subset F_n * H$ be a (possibly infinite) system of equations with constants from H. Then there exists a single equation $f \in F_n * H$ such that $V_H(S) = V_H(f)$.

Splitted equations

Def. An equation $f \in F_n * H$ is called splitted if it has the form wh^{-1}, where $w \in F_n$ and $h \in H$. We can write this equation as

$$w(x_1, \ldots, x_n) = h.$$

Finite systems of equations $\to$ a single splitted equation

Thm B. (Bogo, 2019) Let H be a clean acylindrically hyperbolic group and let $S \subset F_n * H$ be a finite system of equations with constants from H. Then

(1) There exist a natural $k \geqslant n$ and a single splitted equation $f \in F_k * H$ of the form $f_1 f_0$, where $f_1 \in F_k$ and $f_0 \in H$ such that the following two properties are satisfied:

(a)

$$\mathbf{pr}_n\big(V_H(f)\big) = \bigcup_{\alpha \in \mathbb{Z}} V_H(S)^{f_0^\alpha}.$$

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$$\mathbf{pr}_n\big(V_H(f)\big) = \bigcup_{\alpha \in \mathbb{Z}} V_H(S)^{f_0^{\alpha}}.$$

(b) For any overgroup G of the group H we have
$$\mathbf{pr}_n\big(V_G(f)\big) \supseteq \bigcup_{\alpha \in \mathbb{Z}} V_G(S)^{f_0^{\alpha}}.$$

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(2) There exist a natural $k \geqslant n$ and two splitted equations $f, g \in F_k * H$ such that

$$V_H(S) = \mathbf{pr}_n\big(V_H(f)\big) \cap \mathbf{pr}_n\big(V_H(g)\big).$$

Algebraically and verbally closed subgroups. Retracts

Def. Let H be a subgroup of a group G.

(a) The subgroup H is called algebraically closed in G if for any finite system of equations

$$S = \{W_i(x_1, \ldots, x_n; H) = 1 \mid i = 1, \ldots, m\}$$

with constants from H the following holds:
If S has a solution in G, then it has a solution in H.

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(b) The subgroup H is called verbally closed in G if for any word $W \in F(X)$ and any element $h \in H$ the following holds:
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If the equation

$$W(x_1, \dots, x_n) = h$$

has a solution in G, then it has a solution in H.

(c) The subgroup H is called a retract of G if there is a homomorphism $\varphi : G \to H$ such that $\varphi|_H = \mathrm{id}$.

Algebraically closed / verbally closed

Let $H \leqslant G$. When the following statements are equivalent?

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H is f.g., G is f.p.

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Myasnikov and Roman'kov, 2014

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\end{tikzcd}$$

Myasnikov and Roman'kov, 2014 Bogopolski, 2018

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$\uparrow$ solves Problem 5.2 from A.G. Myasnikov, V. Roman'kov,
Verbally closed subgroups of free groups,
J. of Group Theory, **17**, no. 1 (2014), 29-40.

Theorem C fails for non-clean acylindrically hyperbolic groups

Example: Consider two copies of the dihedral group D_4:

$$\begin{aligned} A = & \ \langle a, b \mid a^4 = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle, \\ B = & \ \langle c, d \mid c^4 = 1, d^2 = 1, d^{-1}cd = c^{-1} \rangle. \end{aligned}$$

Let F be a nonabelian free group. We define

$$\begin{aligned} G = & \ F \times (A \underset{a^2=c^2}{\times} B) = (F \times A) \underset{a^2=c^2}{\times} B, \\ H = & \ F \times A. \end{aligned}$$

Clearly, H is hyperbolic and is not clean, and the following holds:

(a) H is verbally closed in G;

(b) H is not a retract of G;

(c) H is not algebraically closed in G.

Classification of elements in acylindrically hyperbolic groups

Def. Let G be a group with a fixed generating set X and let $g \in G$.
g is elliptic if each (equiv. any) orbit of $\langle g \rangle$ in $\Gamma(G, X)$ is bounded.
g is loxodromic if for some $\lambda \geqslant 1, \epsilon \geqslant 0$

$$d(1, g^n) \geqslant \frac{1}{\lambda}|n| - \epsilon.$$

Thm. (Bowditch) Let G be an acylindrically hyperbolic group with respect to a generating set X. Then any element $g \in G$ acts either elliptically or loxodromically on $\Gamma(G, X)$.

About elliptic and loxodromic elements in acylindrically hyperbolic groups

Claim. Let G be an acylindrically hyperbolic group with respect to a generating set X. Let $g \in G$.

• If g is elliptic, then there exists $u \in G$ such that

$$u^{-1}\langle g\rangle u \subseteq B_1(4\delta + 1).$$

• If g is loxodromic, then $\langle g\rangle$ is contained in a unique maximal virtually cyclic subgroup. This subgroup, denoted by $E_G(g)$, is called the elementary subgroup associated with g.
It can be described as

$$\begin{aligned} E_G(g) &= \{f \in G \mid \exists n \in \mathbb{N} : f^{-1}g^n f = g^{\pm n}\} \\ &= \{f \in G \mid \exists k, m \in \mathbb{Z} \setminus \{0\} : f^{-1}g^k f = g^m\}. \end{aligned}$$

Some equations in acylindrically hyperbolic groups

Thm D. (Bogo, 2018) Let G be an acylindrically hyperbolic group with respect to a generating set X. Suppose that $a, b \in G$ are two non-commensurable loxodromic elements (with respect to X) such that $E_G(a) = \langle a \rangle$ and $E_G(b) = \langle b \rangle$.

Then there exists a number $\ell = \ell(a, b) \in \mathbb{N}$ such that for all $n, m \in \ell\mathbb{N}$, $n \neq m$, the equation

$$x^n y^m = a^n b^m$$

is perfect, i.e. any solution of this equation in G is conjugate to (a, b) by a power of $a^n b^m$.

Uniform divergence of quasi-geodesics determined by loxodromic elements in acylindrically hyperbolic groups

Thm E1. (Bogo, 2018) Let G be an acylindrically hyperbolic group with respect to a generating set X. Then there exists a constant $N_0 > 0$ such that for any loxodromic (with respect to X) elements $c, d \in G$ with $E_G(c) \neq E_G(d)$ and for any $n, m \in \mathbb{N}$ we have that

$$|c^n d^m|_X > \frac{\min\{n, m\}}{N_0}.$$

How the quasi-geodesics determined by loxodromic elements avoid balls around the elliptic elements

Thm E2. (Bogo, 2018) Let G be an acylindrically hyperbolic group with respect to a generating set X. Then there exists a constant $N_1 > 0$ such that for any loxodromic (with respect to X) element $c \in G$, for any elliptic element $e \in G \setminus E_G(c)$, and for any $n \in \mathbb{N}$ we have that

$$|c^n e|_X > \frac{n}{N_1}.$$

Jump aside: What happens if two worlds meet

Thm I. (Bogo and Corson, 2019) Let G be an AH-group with respect to a generating set X. Then for any homomorphism $\varphi : \mathrm{HEG} \to G$, there exists a natural number n such that

$$\varphi(\mathrm{HEG}^n) \leqslant \mathrm{Ell}(G, X).$$

Thm J. (Bogo and Corson, 2019) Let H be a topological group which is either completely metrizable or locally compact Hausdorff. Let G be an AH-group with respect to a generating set X. Then for any abstract group homomorphism $\varphi : H \to G$ there exists an open neighborhood V of identity 1_H such that

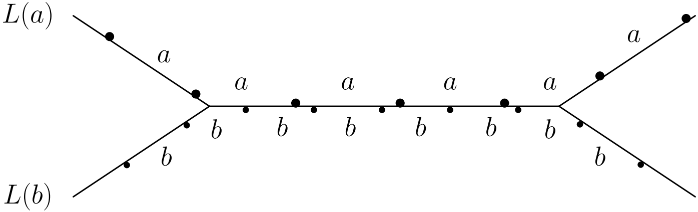
$$\varphi(V) \leqslant \mathrm{Ell}(G, X).$$

1001st proof that HEG is not free

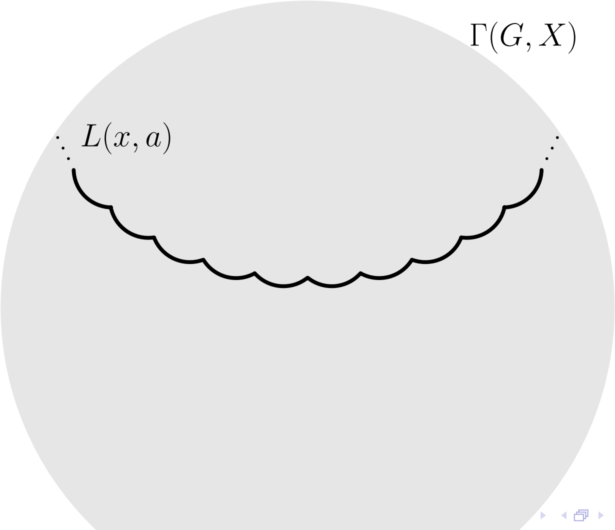
- Each free group F admits a universal acylindrical action on a hyperbolic space.
- HEG dies not admit such an action.

Periodicity theorem for free groups

Claim (Folklore; used by Adian)
Let $a, b \in F(X)$ be two cyclically reduced words. If the bi-infinite words $L(a) = \ldots aaa \ldots$ and $L(b) = \ldots bbb \ldots$ have a common subword of length at least $2\max\{|a|, |b|\}$, then a and b are commensurable.



A quasi-geodesic $L(x, a)$ in the Cayley graph $\Gamma(G, X)$, where $x \in G$, $a \in Lox(G, X)$



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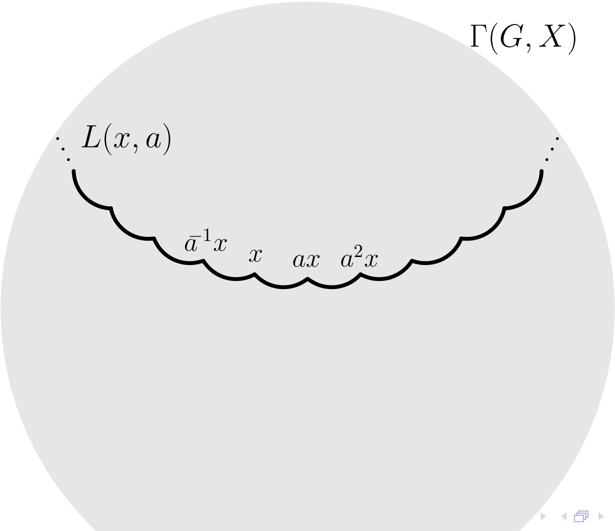


Definition of the set $SLox(G, X)$

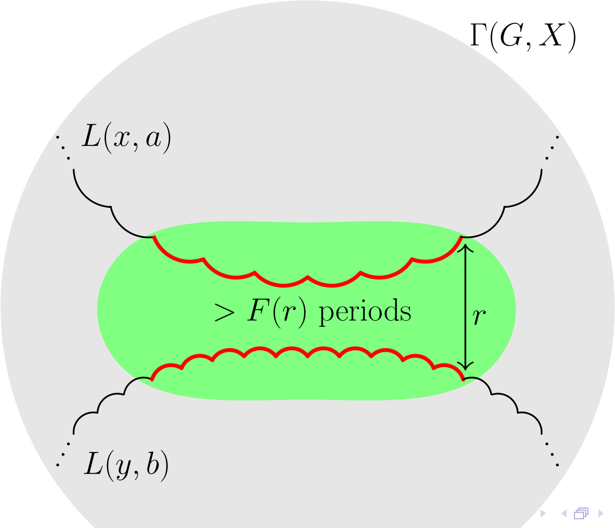
Def. Given a group G and a generating set X of G, we denote by

$$SLox(G, X)$$

the set of of all loxodromic elements of G with respect to X that are shortest in their conjugacy classes.

A periodicity function F

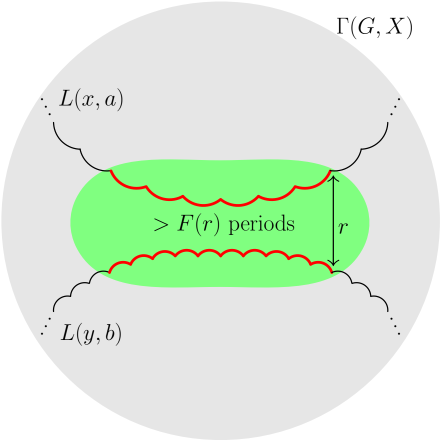
For which groups G and generating sets X, there exists a function $F : \mathbb{R} \to \mathbb{R}$ such that for any $a, b \in SLox(G, X)$, if we have the situation as in the figure below, then a, b are commensurable?



First periodicity theorem for AH-groups

Thm F1. (Bogo, 2018) Suppose that G is an AH-group w.r.t. X. Then there exists a function $\mathbf{F} : \mathbb{R} \to \mathbb{R}$ such that for any two quasi-geodesics $L(x, a)$ and $L(y, b)$, where $a, b \in SLox(G, X)$,

if some segments $p \subset L(x, a)$ and $q \subset L(y, b)$ both contain at least $\mathbf{F}(r)$ periods, where $r = d_{\mathrm{H}}(p, q)$, then a and b are commensurable.



Stable norms

Def. The stable norm of a loxodromic element $g \in G$ with respect to X is the real number defined as

$$||g||_X = \lim_{n \to \infty} \frac{|g^n|_X}{n}.$$

Claim.

(1) $|g|_X \geqslant ||g||_X = \inf_{n \in \mathbb{N}} \frac{|g^n|_X}{n}$.

(2) $||y^{-1}gy||_X = ||g||_X$ for any $y \in G$.

(3) $||g^k||_X = |k| \cdot ||g||_X$ for any $k \in \mathbb{Z}$.

Injectivity radius $\mathbf{inj}(G, X)$

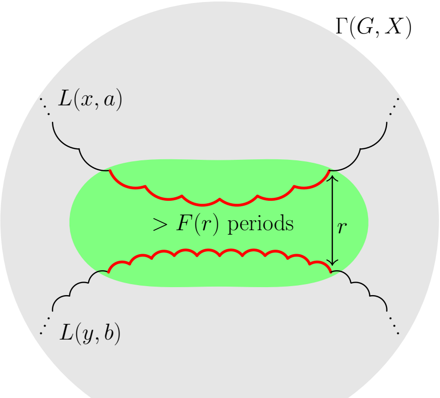
Thm. (Bowditch) Let X be a generating set of a group G.
If the Cayley graph $\Gamma(G, X)$ is hyperbolic and G acts acylindrically on $\Gamma(G, X)$, then each element of G is either elliptic or loxodromic with respect to X and

$$\mathbf{inj}(G, X) := \inf_{g \in Lox(G,X)} ||g|| > 0.$$

First periodicity theorem for AH-groups

Thm F1. (Bogo, 2018) Suppose that G is an AH-group w.r.t. X. Then there exists a function $\mathbf{F} : \mathbb{R} \to \mathbb{R}$ such that for any two quasi-geodesics $L(x, a)$ and $L(y, b)$, where $a, b \in SLox(G, X)$,

if some segments $p \subset L(x, a)$ and $q \subset L(y, b)$ both contain at least $\mathbf{F}(r)$ periods, where $r = d_{\mathrm{H}}(p, q)$, then a and b are commensurable.

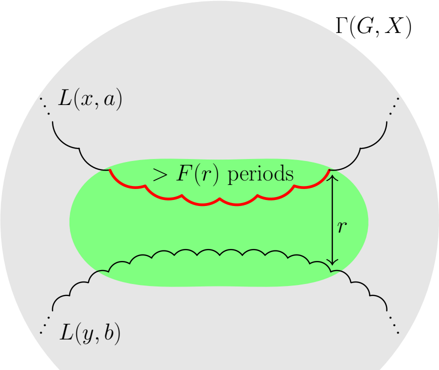


$$\mathbf{F}(r) = \frac{2r}{\mathbf{inj}(G, X)} + C,$$

where $C = C(G, X)$ is a constant.

Second periodicity theorem for AH-groups

Thm F2. (Bogo, 2018) Suppose that G is an AH-group w.r.t. X, and let $\mathbf{F} : \mathbb{R} \to \mathbb{R}$ be the fuction as above. Then for any two quasi-geodesics $L(x,a)$ and $L(y,b)$, where $a, b \in SLox(G,X)$ and $|a|_X \geqslant |b|_X$, and any $r \in \mathbb{R}$
if some segment $p \subset L(x,a)$ contains at least $\mathbf{F}(r)$ periods and lies in the r-neighborhood of $L(y,b)$, then a, b are commensurable. Moreover, there exist $s, t \neq 0$ such that $(x^{-1}y)b^s(y^{-1}x) = a^t$.



$$\mathbf{F}(r) = \frac{2r}{\mathbf{inj}(G,X)} + C$$

Test elements (definition)

The following concept was studied by Nielsen, Zieschang, Rips, Dold, Spilrain, Turner and otheres before it was defined explicitly by O'Neill and Turner (2000).

Def. Given a group G, an element $g \in G$ is called a test element if any endomorphism $\varphi : G \to G$ for which $\varphi(g) = g$ is an automorphism of G.

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Examples. The following elements are test elements:

- $[x_1, x_2]$ in the free group $F(x_1, x_2)$ (Nielsen).
- $[x_1, x_2] \cdot \ldots \cdot [x_{n-1}, x_n]$ in $F(x_1, \ldots, x_n)$ if n is even (Zieschang).
- $x_1^k x_2^k \ldots x_n^k$ $(k \geqslant 2)$ in $F(x_1, \ldots, x_n)$ (Zieschang)

Test elements and retracts

Def. Given a group G, an element $g \in G$ is called a test element if any endomorphism $\varphi : G \to G$ for which $\varphi(g) = g$ is an automorphism of G.

Obs. Test elements of G lie outside of proper retracts of G.

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Thm. (Turner, 1996) An element g of a free group F_n is a test element if and only if g is not contained in a proper retract of F_n.

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Thm. (Turner, 1996) An element g of a free group F_n is a test element if and only if g is not contained in a proper retract of F_n.

Thm. (Groves, 2012) An element g of a torsion-free hyperbolic group G is a test element if and only if g is not contained in a proper retract of G.

Test words (new definition)

Def. (Bogo, 2018) Let G be a group and let $a_1, \dots, a_k$ be some elements of G. A word $W(x_1, \dots, x_k)$ from F_k is called an $(a_1, \dots, a_k)$-test word if any solution of the equation

$$W(x_1, \dots, x_k) = W(a_1, \dots, a_k) \qquad (\star)$$

in G is usual, i.e., any solution $(b_1, \dots, b_k)$ has the form

$$(b_1, \dots, b_k) = (a_1, \dots, a_k)^U,$$

where U is a power of the element on the right side of $(\star)$.

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where U is a power of the element on the right side of $(\star)$.

Remark 1. If $G = \langle a_1, \ldots, a_k \rangle$ and $W(x_1, \ldots, x_k)$ is as above, then the element $W(a_1, \ldots, a_k)$ is a test element in the sense of Turner. Moreover, any endomorphism of G fixing this element is a conjugation.

Why test words are good

Def. (Bogo, 2018) Let G be a group and let $a_1, \ldots, a_k$ be some elements of G. A word $W(x_1, \ldots, x_k)$ from F_k is called an $(a_1, \ldots, a_k)$-test word if any solution of the equation

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Remark 2.
1) If $G = \langle a_1, \ldots, a_k \rangle$, then any outer automorphism of G is completely determined by its values on a single element $g = W(a_1, \ldots, a_k)$, where $W(x_1, \ldots, x_k)$ is as above.

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Compare (S. Ivanov): There exist two elements $g_1, g_2 \in F_n$ such that any monomorphism of F_n is completely determined by its value on g_1, g_2.

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2) Test words are used in the proof of main Theorems A and B.

Existence of test words

Def. (Bogo, 2018) Let G be a group and let $a_1, \dots, a_k$ be some elements of G. A word $W(x_1, \dots, x_k)$ from F_k is called an $(a_1, \dots, a_k)$-test word if any solution of the equation

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in G is usual (see above).

Metha-Thm G. (Bogo, 2018)
If G is a clean AH-group, then for any generic finite set of elements $a_1, \dots, a_k$ in G there exists an $(a_1, \dots, a_k)$-test word.

Special elements in acylindrically hyperbolic group

Def. (Bogo, 2019)
Suppose that G is an acylindrically hyperbolic group.

(a) An element $g \in G$ is called special if there exists a generating set X of G such that
- G is acylindrically hyperbolic with respect to X,
- g is loxodromic with respect to X, and
- $E_G(g) = \langle g \rangle$.

In this case g is called special with respect to X.

(b) Elements $g_1, \ldots, g_k \in G$ are called jointly special if there exists a generating set X of G such that each g_i is special with respect to X.

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(b) Elements $g_1, \ldots, g_k \in G$ are called jointly special if there exists a generating set X of G such that each g_i is special with respect to X.

Example. Any set of root-free elements of F_n is jointly special.

Existence of special elements

Rem. A non-clean AH-group does not contain special elements.

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Existence of special elements

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Lem. (Osin) Any clean AH-group contains a special element.

Lem. (Bogo, 2019) Let G be an clean AH-group. Then there are two special elements $a, g \in G$ such that for any $k \in \mathbb{N}$ the coset $a\langle g \rangle$ contains k non-commensurable and jointly special elements.

Generation by jointly special and non-commensurable elements

Prop H. (Bogo, 2018) Let G be an AH-group and H a finitely generated subgroup of G. If H contains at least one special element of G, then H can be generated by a finite set of elements, which are pairwise non-commensurable and jointly special in G.

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Cor H1. (Bogo, 2018) Any finitely generated clean AH-group G can be generated by by a finite set of elements, which are pairwise non-commensurable and jointly special in G.

Existence of test words

Thm G1. (Bogo, 2018) Let G be an AH-group. Suppose that $a_1, \dots, a_k \in G$, where $k \geqslant 1$, are jointly special and pairwise non-commensurable. Then there exists an $(a_1, \dots, a_k)$-test word $\mathcal{U}_k(x_1, \dots, x_k)$.

Moreover, one can choose $\mathcal{U}_k(x_1, \dots, x_k)$ so that the elements $a_1, \dots, a_k$ together with $\mathcal{U}_k(a_1, \dots, a_k)$ are jointly special and pairwise non-commensurable.

Existence of test words

Thm G2. (Bogo, 2018) Let G be an AH-group. Suppose that $a_1, \dots, a_k \in G$, where $k \geqslant 3$, are jointly special and pairwise non-commensurable. Then there exists an $(a_1, \dots, a_k, \mathbf{1}^{k-2})$-test word $W_k(x_1, \dots, x_k, y_3, \dots, y_k)$.

Moreover, one can choose this test word so that the elements $a_1, \dots, a_k$ together with $W_k(a_1, \dots, a_k, \mathbf{1}^{k-2})$ are jointly special and pairwise non-commensurable.

Proof of Theorem G1

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Proof.
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Proof.
$\underline{k = 1}$. Take $\mathcal{U}_1 = x_1$.

$\underline{k = 2}$. Take $\mathcal{U}_2 = x_1^n x_2^m$ with n, m from Theorem D.

Equation $x^n y^m = a^n b^m$ in AH-groups

Thm D. (Bogo, 2018) Let G be an AH-group. Suppose that $a, b \in G$ are two non-commensurable jointly special elements. Then there exists a number $\ell = \ell(a, b) \in \mathbb{N}$ such that for all $n, m \in \ell\mathbb{N}$, $n \neq m$, any solution of the equation

$$x^n y^m = a^n b^m$$

in G is conjugate to (a, b) by a power of $a^n b^m$.

Proof of Theorem G1

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Proof.
$\underline{k = 1}$. Take $\mathcal{U}_1 = x_1$.

$\underline{k = 2}$. Take $\mathcal{U}_2 = x_1^n x_2^m$ with n, m from Theorem D.

$\underline{k = 3}$. Do not take $\mathcal{U}_3 = (x_1^n x_2^m)^s x_3^t$. Entangle:

$$\mathcal{U}_3 = \left(\left(x_1^k x_2^l\right)^p \, \left(x_2^n x_3^m\right)^q \right)^u \, \left(x_2^s x_3^t \right)^v$$

Proof of Theorem G2

Thm G2. (Bogo, 2018) Let G be an AH-group. Suppose that $a_1, \ldots, a_k \in G$, where $k \geqslant 3$, are jointly special and pairwise non-commensurable. Then there exists an $(a_1, \ldots, a_k, \mathbf{1}^{k-2})$-test word $W_k(x_1, \ldots, x_k, y_3, \ldots, y_k)$.

Proof. The proof is more tricky.
For $k = 3$ the $(a_1, a_2, a_3, \mathbf{1})$-test word has the form

$$\left(\left(x_1^k x_2^l\right)^p \left(x_2^n x_3^m\right)^q \right)^u \left(x_2^s (x_3 x_4)^t\right)^v$$

with exponents depending on a_i's.

Systems of equations $\to$ a single equation

Thm A. (Bogo, 2018) Let H be a clean acylindrically hyperbolic group and let

$$S : \begin{cases} s_1(x_1, \dots, x_n; H) = 1, \\ \dots \\ s_k(x_1, \dots, x_n; H) = 1 \end{cases}$$

be a finite system of equations with constants from H. Then there exists a single equation $f(x_1, \dots, x_n; H) = 1$ with $V_H(S) = V_H(f)$.

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Proof. 1) Take jointly special and non-commen. $a_1, \dots, a_{k+2} \in H$.

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Proof. 1) Take jointly special and non-commen. $a_1, \ldots, a_{k+2} \in H$.

2) Let $W(z_1, \ldots, z_{2k+2})$ be an $(a_1, \ldots, a_{k+2}, \underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{k})$-test word.

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3) The desired single equation is

$$W(a_1, \ldots, a_{k+2}, \underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{k}) = W(a_1, \ldots, a_{k+2}, s_1, \ldots, s_k).$$

Finite systems of equations $\to$ a single splitted equation

Thm B. (Bogo, 2019) Let H be a clean acylindrically hyperbolic group and let $S \subset F_n * H$ be a finite system of equations with constants from H. Then there exist a natural $k \geqslant n$ and a single splitted equation $f \in F_k * H$ of the form $f_1 f_0$, where $f_1 \in F_k$ and $f_0 \in H$ such that the following holds:

(a)
$$\mathbf{pr}_n\big(V_H(f)\big) = \bigcup_{\alpha \in \mathbb{Z}} V_H(S)^{f_0^\alpha}.$$

(b) For any overgroup G of the group H we have

$$\mathbf{pr}_n\big(V_G(f)\big) \supseteq \bigcup_{\alpha \in \mathbb{Z}} V_G(S)^{f_0^\alpha}.$$

Algebraic closedness / verbal closedness

Thm C. (Bogo, 2018) If H is a clean acylindrically hyperbolic group and G is an arbitrary overgroup of H, then the following properties are equivalent:

(1) H is algebraically closed in G.
(2) H is verbally closed in G.

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Proof. (2) $\Rightarrow$ (1). We shall prove the implication

$$V_G(S) \neq \emptyset \;\Rightarrow\; V_H(S) \neq \emptyset.$$

Let f be a single splitted equation as in Theorem B.

$$V_G(S) \neq \emptyset \overset{(b)}{\Rightarrow} V_G(f) \neq \emptyset \overset{(2)}{\Rightarrow} V_H(f) \neq \emptyset \overset{(a)}{\Rightarrow} V_H(S) \neq \emptyset.$$

Existential closedness

Def. A subgroup H of a group G is called existentially closed in G if for any finite system S of equations and inequalities

$$\begin{aligned} u_i(x_1, \ldots, x_k; H) &= 1 \quad (i = 1, \ldots, n), \\ v_j(x_1, \ldots, x_k; H) &\neq 1 \quad (j = 1, \ldots, m) \end{aligned}$$

with coefficients in H the following holds: if S has a solution in G, then it has a solution in H.

Discrimination and separation

The following definition is due to

G. Baumslag, A. Myasnikov, and V. Remeslennikov, *Algebraic geometry over groups I. Algebraic sets and ideal theory*, J. Algebra, 1999.

Def. (BMR)

1) If H is a subgroup of G, the expression $H \leqslant G$ is called an extension of H to G.

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1) If H is a subgroup of G, the expression $H \leqslant G$ is called an extension of H to G.

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2) An extension $H \leqslant G$ is called discriminating if for any finite subset $K \subseteq G$ there exists a retraction $\varphi : G \to H$ that is injective on K.

3) An extension $H \leqslant G$ is called separating if for any nontrivial element $g \in G$ there exists a retraction $\varphi : G \to H$ such that $\varphi(g) \neq 1$.

Existential closedness/ discrimination / separation

Let $H \leqslant G$. When the following statements are equivalent?

(1) H is existentially closed in G.

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\textbf{exist. closed} & & \textbf{discriminat.} \arrow[ll, bend left, "\text{always}"]
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Myasnikov and Roman'kov, 2014

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\textbf{exist. closed} \arrow[r, bend left, "\substack{H \text{ is f.g., } G \text{ is f.p. or} \\ H \text{ is e.N., } G \text{ is f.g. over } H}"] & \textbf{discriminat.} \arrow[l, bend left, "\text{always}"]
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\end{tikzcd}$$

Myasnikov and Roman'kov, 2014 Bogopolski, 2019

Discrimination and separation

Thm H. (Bogo, 2019) Suppose that H is a clean AH-group and G is an overgroup of H. Then the extension $H \leqslant G$ is discriminating if and only it is separating.

Existential closedness for subgroups of relatively hyperbolic groups

Cor H1. (Bogo, 2019) Let G be a clean relatively hyperbolic group with respect to a finite collection of proper finitely generated equationally noetherian subgroups. Then for any subgroup H of G the following statements are equivalent.

(1) H is existentially closed in G.

(2) The extension $H \leqslant G$ is discriminating.

(3) The extension $H \leqslant G$ is separating.

In particular, these statements are equivalent for any subgroup H of a clean hyperbolic group G.

$\mathfrak{S}$-closedness (preparatory notations)

• Let $\mathfrak{A}$ be some class of algebraic structures with the same signature *Sign*. For example, $\mathfrak{A}$ is a class of all hyperbolic groups with the signature $Sign = \{\cdot, {}^{-1}, 1\}$.

• Let *Lang* be the language consisting of
- the elements of *Sign*,
- the logical symbols $=$, $\neg$, $\bigwedge$, $\bigvee$, $\exists$, $\forall$,
- the variables $x_1, x_2, \ldots,$ and the punctuation signs.

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Def. Suppose that ϕ is a first-order formula in the language *Lang*. A variable x in ϕ is called free in ϕ if neither $\forall x$ nor $\exists x$ occur in ϕ.

We denote ϕ by $\phi(x_1, \dots, x_n)$ if and only if $x_1, \dots, x_n$ are all free variables of ϕ.

$\mathfrak{S}$-closedness (definition)

Def (B.H. Neumann)
- Let $\mathfrak{A}$ be a class of structures with the same signature *Sign*.
- Let A be a structure from the class $\mathfrak{A}$.
- Let $\mathfrak{S}$ be a set of formulas in the language *Lang*.

The structure A is called *$\mathfrak{S}$-closed in $\mathfrak{A}$* if for any formula $\phi(x_1, \ldots, x_n) \in \mathfrak{S}$, any elements $a_1, \ldots, a_n \in A$, and any structure $B \in \mathfrak{A}$ containing A if $\phi(a_1, \ldots, a_n)$ holds

$$B \models \phi(a_1, \ldots, a_n) \Rightarrow A \models \phi(a_1, \ldots, a_n).$$

In the case, where H is a substructure of a structure G and H is $\mathfrak{S}$-closed in the class $\{H, G\}$, we simplify notation by saying that *H is $\mathfrak{S}$-closed in G*.

Some subsets of the set of all formulas

- Φ is the set of all formulas in the language associated with the signature of groups.

- Φ_0 is the subset of Φ consisting of all formulas without free variables (such formulas are called sentences).

- $\exists$ is the subset of Φ consisting of all existential formulas, i.e., of the formulas which have the form $\exists x_1 \dots \exists x_n \, \phi(x_1, \dots, x_n, x_{n+1}, \dots, x_k)$, where $\phi \in \Phi$ is a quantifier free formula.

- $\exists^+$ is the subset of Φ consisting of all positive existential formulas, i.e., of the existential formulas without the negation symbol.

- $\mathcal{V} = \bigcup\limits_{n \in \mathbb{N}} \{\exists x_1 \dots \exists x_n : w(x_1, \dots, x_n) = x_{n+1} \mid w \in F_n\}$.

Vocabulary

Let $H \leqslant G$.

elementary embedded	Φ-closed	$\Rightarrow \mathbf{Th}(H) = \mathbf{Th}(G)$
existentially closed	$\exists$-closed	$\Rightarrow \mathbf{Th}_\exists(H) = \mathbf{Th}_\exists(G)$
algebraically closed	$\exists^+$-closed	
verbally closed	$\mathcal{V}$-closed	

Thm. (Sela; Kharlampovich and Myasnikov) For $n \geqslant k \geqslant 2$, the subgroup $F(x_1, \ldots, x_k)$ is elementarily embedded in $F(x_1, \ldots, x_n)$.

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Thm. (Perin, Sela) Let H be a subgroup of a torsion-free hyperbolic group G. H is elementarily embedded in G if and only if G admits the structure of a hyperbolic tower over H (in particular, H is a retract of G).

Comparatione

Let H be a subgroup of a torsion-free hyperbolic group G. Then

H is verbally closed in G $\Updownarrow$ H is a retract of G	H is existentially closed in G $\Updownarrow$ For any nontrivial $g \in G$ there exists a retraction $\varphi : G \to H$ with $\varphi(g) \neq 1$	H is elementary embedded in G $\Updownarrow$ G is a hyperbolic tower over H

Problems for future

1. Describe solutions of equations with high exponents in the class of acylindrically hyperbolic groups.

2. Characterise elementary embedding in the class of acylindrically hyperbolic groups.

THANK YOU!